

# Quantum Field Theory I

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## **Abstract**

These are the notes for the first quarter course in Quantum Field Theory (QFT) at UC Santa Cruz. Following [1], it focuses exclusively on scalar fields, introduces path integrals almost immediately, and uses the mostly plus metric convention.

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## What is quantum field theory?

We don’t know. There is no mathematically precise definition of quantum field theory in general. If you’re a mathematician, this subject will drive you up the wall. At the physicist’s level of rigor, the best logical construction of quantum field theory in my opinion is done by Steven Weinberg, in his book *The Quantum Theory of Fields Vol. 1* [2]. But it is somewhat impenetrable on a first read, so I would not suggest starting there.

Let’s discuss, colloquially, what quantum field theory (QFT) is. QFT is a general framework for the quantization of classical field theory. High energy physicists often care about it in the context where there is also Lorentz invariance, since they are interested in the fundamental theories that govern our universe, which are Lorentz invariant as far as we can tell. Condensed matter physicists also use this framework, but are happy to leave out Lorentz invariance since it is not useful in many situations at low energies. So, while QFT solves the problem of unifying quantum mechanics and special relativity, as you may have heard, it is actually even more powerful than this. It can treat all sorts of non-relativistic cases of physical interest, e.g. the inflationary phase of our universe or tabletop experiments. We will, however, focus exclusively on the Lorentz-invariant case.

One of the surprising things that happens in a quantum field theory class – it definitely confused me for years – is what in the world happened to quantum mechanics, which I thought I knew well. We are taught quantum mechanics as computing eigenvectors and eigenvalues. Wavefunctions are everywhere. But if you open most QFT textbooks, spotting the wavefunction is about as difficult as finding Waldo. This is partly because the *goal* of many of these textbooks is to compute perturbative processes in the Standard Model of particle physics. Then the relevant object is a sort of overlap of eigenvectors that in the end produces some probability amplitudes. But the wavefunction, like Waldo, is there, hiding, all the time. It is good to remember this and ground yourself in ordinary quantum mechanics if you feel yourself getting lost.

## Things you'll need to know

One of the difficult parts about learning quantum field theory is the large background you will need and new physical ideas that you will have to understand. If you don't have familiarity with the old stuff, and if you can't identify the new physical ideas as they appear, then you are at risk of losing the forest for the trees. When I learned the subject, I couldn't tell what was new, what was old, what I should have already known, etc., so it was a struggle for me. So hopefully this list will help a bit in organizing the course topics in your head.

Here is some of the background that will make your life easier if you review:

1. Complex analysis: residue theorem etc.
2. Green's function solution of differential equations
3. Classical field theory and variational calculus
4. Index notation and Einstein summation convention from relativity

Some of the new ideas that will come your way include the following:

1. Ultraviolet divergences
2. Infrared divergences
3. Representation theory, in particular representations of the Lorentz group; this is to deal with non-scalar particles like electrons and photons.
4. Path integrals
5. Gauge invariance and gauge theory

The first two topics have to do with the infinities that plague QFT, which you may have heard of. They are very confusing and dealing with them often feels ad hoc. We will try to handle them carefully and isolate their appearance.

## Conventions

Here is a list of some of the conventions and formulas used in these notes:

- $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ , i.e. we use the “mostly plus” metric which [1] and [2] use (and most others do not).
- Latin indices  $i, j$  etc. run over spatial coordinates while Greek indices  $\mu, \nu$  etc. run over space and time.
- $\Delta := \nabla^2 := \delta^{ij} \partial_i \partial_j = \partial_x^2 + \partial_y^2 + \partial_z^2$
- $\square := \partial^2 := \eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$
- Sometimes  $\partial_x^2$  will be used to mean  $\square_x$  when acting on a function of two spacetime positions  $G(x, y)$ . In this context  $\partial_x^2$  is simply the d'Alembertian  $\square_x$  acting on the first argument.
- In non-relativistic quantum mechanics we have (with  $\hbar = 1$ )

$$[x, p] = i, \quad \langle x|x'\rangle = \delta(x - x'), \quad \langle p|p'\rangle = \delta(p - p'), \quad \langle x|p\rangle = \frac{e^{ipx}}{\sqrt{2\pi}},$$

$$\mathbf{1} = \int dp |p\rangle \langle p| = \int dx |x\rangle \langle x|,$$

$$\delta(x - x') = \langle x|x'\rangle = \int dp \langle x|p\rangle \langle p|x'\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')}, \quad \mathbf{1} = \int_{-\infty}^{\infty} dx \delta(x - x')$$

Heisenberg picture:  $O(t) = e^{iHt} O(0) e^{-iHt}$ , Schrödinger picture:  $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$

$$\implies \langle \psi | O(t) | \psi \rangle_{\text{Heisenberg}} = \langle \psi(t) | O | \psi(t) \rangle_{\text{Schrödinger}}$$

## Why these notes?

These notes are a blend of many textbooks and people that I learned the subject from. Many of today's standard textbooks are aimed toward computing processes in the Standard Model of particle physics as quickly as possible. While this is important for many reasons – diving into research on particle phenomenology one of them – it is not the optimal path for learning the subject. And indeed, condensed matter physicists use QFT all the time and are less interested in this perspective.

One of the problems with the usual approach is that there are too many novel things appearing all at once. Confronting the Standard Model forces you to immediately deal with the representation theory of the Lorentz group, fermions, and massless gauge fields and their ensuing infrared divergences. This comes on top of one of the deepest aspects of QFT – ultraviolet divergences and renormalization – and it is very easy to get lost. The fact that the general mathematical machinery of QFT, including dealing with ultraviolet divergences and renormalization, can be studied simply in scalar field theory is an opportunity few textbooks take advantage of.

Another problem with the usual approach – although this is more controversial – is the postponing of path integral methods until very late in the course. This seems to be simply anachronistic. It is both the quickest way to calculating Feynman diagrams and a powerful tool for many modern applications of QFT.

On both these counts, the textbook by Srednicki [1] is wonderful, since it focuses on scalar fields and the path integral from the start. Many students, however, have felt that the pedagogy in Srednicki is a bit minimalistic. This is where the textbooks by Coleman [3] and Schwartz [4] shine, although the former is somewhat out of date by today.

It seems to me that the ideal modern textbook would have the topics mostly ordered according to Srednicki but with the pedagogy of Coleman and Schwartz. This is what I have tried to do in these notes. Of course, everyone's perspective on this is different, and I have surely missed the mark in many places. There are some common stumbling blocks in QFT which I tried to disambiguate by the following

- Several appendices are included which cover background that students need but are often missing, e.g. retarded vs advanced Green's functions and how they relate to the chosen boundary conditions.
- Feynman diagrams are introduced both in the context of classical differential equations and ordinary one-dimensional integrals to demystify them.
- The LSZ formula is punted as far back as possible, since the subject of QFT is much

more than calculating scattering amplitudes.

- The wavefunction makes an appearance in a few places so that the connection to quantum mechanics is a little clearer.
- Functions and functionals are carefully disambiguated from the start.

After a first course, to understand the subject much more deeply, I recommend to everyone that they tackle Weinberg's brilliant textbook [2]. I have never met someone who has done well with this book on a first read. But it is the closest any book has come to a logical construction of the subject (for a physicist), very carefully building it from the ground up with basic physical principles. This book was definitely a revelation for me.

## 1 Historical progression

The route to QFT was a mess. Given your coursework, you might expect that first quantum mechanics and special relativity were developed, and then some people thought really hard and figured out how to unify them. Instead, the history of QFT is very tied into the history of quantum mechanics itself. After all, physicists were trying to understand things like photons, fermions, etc., not so much particles in a box with infinite potential walls. So the two subjects actually developed more in lockstep. Chapter 1 of [2] gives a nice history of the subject; we will not review it here. Like Weinberg, I believe that the historical route can cloud the logical route (and especially does so in the case of this subject). Sections 1.1 - 1.3 of [4] give a select historical progression to motivate QFT, starting from blackbody radiation and the ultraviolet catastrophe. That is another place you can look.

## 2 Attempts at relativistic quantum mechanics

As discussed in the introduction, QFT can be thought of as the quantization of classical fields. Lorentz invariance is not necessary. But in this course we will exclusively focus on Lorentz invariant theories. Then it is natural to ask: why are we forced into introducing the notion of fields when trying to unify quantum mechanics and special relativity? The best treatment of this is in Chapters 2 - 5 of [2]. This is around 160 pages of material before fields are introduced, so we will instead content ourselves with some motivation and then jump straight into the fields.

So let's try a naive attempt at relativistic quantum mechanics. We take the fundamental equation in quantum mechanics, the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle. \quad (2.1)$$

Hats  $\hat{\phantom{x}}$  represent a quantum operator. As you know in quantum mechanics, the physics of the system you care about is all encoded in the Hamiltonian operator  $\hat{H}$ , whose expectation value gives the total energy. A free particle with mass  $m$  has the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} \quad (2.2)$$

where  $\hat{p}_i$  represents the 3-momentum of the particle and  $\hat{p}^2 = \hat{p}_i \hat{p}^i$ . To write the wavefunction in the position basis, we take the overlap of the ket wavefunction with a position eigenvector bra:

$$\psi(x_i, t) \equiv \langle x_i | \psi(t) \rangle. \quad (2.3)$$

Using the position basis representation of the momentum operator, i.e.

$$\hat{p}_i = - \int dx_i |x_i\rangle i\hbar \nabla_i \langle x_i|, \quad (2.4)$$

gives

$$\langle x_i | i\hbar \frac{\partial}{\partial t} |\psi\rangle = \langle x_i | \frac{\hat{p}_i^2}{2m} |\psi\rangle \implies i\hbar \frac{\partial}{\partial t} \psi(x_i, t) = - \frac{\hbar^2}{2m} \nabla^2 \psi(x_i, t). \quad (2.5)$$

This is a non-relativistic equation, because the Hamiltonian that was used was modeled after

the kinetic energy  $E = \frac{1}{2}mv^2 = p^2/(2m)$ . We can instead use the relativistic equation

$$p^\mu p_\mu \equiv m^2 u^\mu u_\mu = -m^2 c^2 = -E^2/c^2 + p^2 \rightarrow \hat{H} = \sqrt{\hat{p}^2 c^2 + m^2 c^4} \quad (2.6)$$

Notice that for  $v \ll c$ , i.e.  $p \ll mc$ , we can expand this as

$$\hat{H} = mc^2 + \frac{\hat{p}^2}{2m} - \frac{1}{8c^2 m^3} \hat{p}^4 + \dots \quad (2.7)$$

The first term is the famous Einstein rest-energy term, the second term is the nonrelativistic term we saw before, but then there is an infinite series of higher order terms in  $\hat{p}^2$ .

In the position basis we have

$$i\hbar \frac{\partial}{\partial t} \psi(x_i, t) = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4} \psi(x_i, t). \quad (2.8)$$

This attempt has two serious problems. The first is that if we expand the square root as before we have terms with arbitrarily high powers of  $\nabla^2$ . This usually signals nonlocality of the theory, since high derivatives of a function can access the value of the function far away, in the sense of a Taylor series. So this would say the time derivative of the wavefunction at a point in spacetime is related to the value of the wavefunction arbitrarily far away. That is nonlocal.

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*Example 1:* We can explicitly calculate the propagator using the square-root Hamiltonian to see that things can travel faster than light. Let's warm up with the quantum-mechanical case,  $\hat{H} = \hat{p}^2/(2m)$ . The propagator is

$$U(t) = \langle \mathbf{x}_2 | e^{-i\hat{H}t} | \mathbf{x}_1 \rangle. \quad (2.9)$$

Inserting a complete set of momentum eigenstates gives

$$U(t) = \int d^3p \langle \mathbf{x}_2 | e^{-i\frac{p^2}{2m}t} | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{x}_1 \rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\frac{p^2}{2m}t + i\mathbf{p} \cdot (\mathbf{x}_2 - \mathbf{x}_1)} \quad (2.10)$$

We switch to spherical coordinates,  $d^3p = p^2 \sin\theta dp d\theta d\phi$  and align the  $z$  axis along  $\mathbf{x}_2 - \mathbf{x}_1$  so that  $\mathbf{p} \cdot (\mathbf{x}_2 - \mathbf{x}_1) = p|\mathbf{x}_2 - \mathbf{x}_1| \cos\theta$ . Doing the  $\theta$  and  $\phi$  integrals gives

$$U(t) = \frac{1}{2\pi^2 |\mathbf{x}_2 - \mathbf{x}_1|} \int_0^\infty dp p \sin(p|\mathbf{x}_2 - \mathbf{x}_1|) e^{-\frac{ip^2 t}{2m}}. \quad (2.11)$$

To evaluate this integral, we can analytically continue  $t$  to have a slight negative imaginary

part  $t \rightarrow t - i\epsilon$  with  $\epsilon > 0$ . This provides a Gaussian suppression, and after doing the integral we can set  $\epsilon \rightarrow 0$  to get

$$U(t) = \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(\mathbf{x}_2 - \mathbf{x}_1)^2/(2t)}. \quad (2.12)$$

This propagator does not vanish for spacelike separated points, so does not respect Lorentz invariance.

If we repeat the calculation with  $\hat{H} = \sqrt{\hat{p}^2 + m^2}$ , the same steps lead to

$$U(t) = \frac{1}{2\pi^2|\mathbf{x}_2 - \mathbf{x}_1|} \int_0^\infty dp p \sin(p|\mathbf{x}_2 - \mathbf{x}_1|) e^{-it\sqrt{p^2 + m^2}}. \quad (2.13)$$

This can be evaluated and is again nonzero for spacelike separation. One way to approximate it is to take a large spacelike separation  $x \equiv |\mathbf{x}_2 - \mathbf{x}_1| \gg t$  and evaluate the integral by stationary phase (see Appendix D.1 for a quick review). Writing the sine as complex exponentials leads to two integrals, with a stationary point  $p = imx/\sqrt{x^2 - t^2}$  for one integral and  $p = -imx/\sqrt{x^2 - t^2}$  for the other integral. Plugging this in gives

$$U(t) \sim \frac{m}{\sqrt{x^2 - t^2}} e^{-m\sqrt{x^2 - t^2}}. \quad (2.14)$$

We again see a nonzero value for the propagator for spacelike separation.

Another problem with the square-root Hamiltonian is that time and space are not treated symmetrically in the resulting equation (2.8): we have a single time derivative on the left and arbitrarily many spatial derivatives on the right.

To solve both problems, let's square the differential operators on both sides of the equation. This leads to

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(x_i, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi(x_i, t). \quad (2.15)$$

This is known as the Klein-Gordon equation. Notice that for  $m = 0$  it is nothing but the wave equation, which describes sound waves, water waves, etc. This differential equation is well-behaved and describes local propagation.

We should be a little more precise about what the symmetry between space and time means. Really we want the equation to be consistent with special relativity. This means that the physics should be the same in all inertial reference frames. An inertial reference frame is

one that does not accelerate. It can be given coordinates  $x^\mu = (x^0, x^i) = (ct, x^i)$  with metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \quad (2.16)$$

This means we are using the Minkowski metric  $\eta_{\mu\nu}$  with sign convention  $(-, +, +, +)$  ([4] uses the sign convention  $(+, -, -, -)$ , favored by particle physicists without the proper respect for time). This metric is used to raise and lower indices, so we have  $x_\mu = \eta_{\mu\nu} x^\nu \implies x_0 = -x^0, x_i = x^i$ . The “physics” in this case will mean the value of the wavefunction. To see that the value of the wavefunction is the same between reference frames, we should show that the equation that governs the wavefunction is a *Lorentz scalar*. Lorentz scalars have the same value in different reference frames. For example, the spatial length of something is not a Lorentz scalar, since lengths contract between reference frames, but the spacetime length as measured by  $-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$  is a Lorentz scalar. When we use index notation, Lorentz scalars are objects with no free indices (the invariant spacetime length can be represented for example as  $x^\mu x_\mu$ , all indices are contracted).

We can see that the Klein-Gordon equation has all indices contracted. For that we remind ourselves of the definition of 4-derivatives,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \frac{\partial}{\partial t}, \nabla_i \right) \quad (2.17)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( -\frac{1}{c} \frac{\partial}{\partial t}, \nabla_i \right) \quad (2.18)$$

We can therefore write the Klein-Gordon equation as

$$(\partial_\mu \partial^\mu - m^2 c^2 / \hbar^2) \psi(x_i, t) = 0. \quad (2.19)$$

Notice that all indices are contracted.  $\partial_\mu \partial^\mu$  is sometimes written as  $\square$ , a relativistic generalization of  $\Delta := \nabla^2$  (get it?  $\square$  has 4 corners and  $\Delta$  has 3.) So the equation governing the wavefunction is Lorentz invariant; success!

An essential aspect of quantum mechanics, however, is that probability is conserved. This means that the norm of the wavefunction, i.e. the integrated probability amplitude  $\int dV \langle \psi(x_i, t) | \psi(x_i, t) \rangle = \int dV |\psi(x_i, t)|^2$  is time-independent (and equal to 1, by normalization of the wavefunction). This is usually done by taking the time derivative of  $|\psi|^2$  and using the Schrödinger equation (and its complex conjugate) to rewrite the expression as the divergence of something usually called the “probability current” (since the time derivative gives  $\psi \partial_t \psi^* + \psi^* \partial_t \psi$  any (real-valued) potential cancels between the two terms after using the Schrödinger equation). We integrate over the volume of spacetime using the divergence

theorem to rewrite it as a boundary term only. But since the wavefunction is required to go to zero at infinity, this boundary term vanishes and we find that the time derivative of the norm of the wavefunction vanishes.

Since our new equation has a second time derivative in it, this procedure does not work in the same way, and in fact the norm of the wavefunction is not preserved in time. That is a disaster!

An attempt to deal with this was made by Dirac, who postulated an equation that is linear in time and space derivatives:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0. \quad (2.20)$$

Since we are still warming up with index notation, let's write this a little more carefully:

$$(i\hbar(\gamma^\mu)_{ab}\partial_\mu - mc\delta_{ab})\psi(x)_b = 0. \quad (2.21)$$

We have *two* sets of indices: the Greek ones e.g.  $\mu, \nu$  refer to 4-vectors and the spacetime symmetry. The Latin ones e.g.  $a, b$  are a new index on the field  $\psi$ . It turns out that in this equation the Latin indices run over 1, 2, 3, 4. Since the Greek indices are contracted, that means this is a  $4 \times 1$  vector equation, i.e. it packages together four equations. This equation is also problematic, as it turns out that the Hamiltonian has negative eigenvalues for every positive eigenvalue! See Section 1 of [1] for details.

So the naive attempts at a relativistic quantum-mechanical equation for a particle seem to fail. But why do we suddenly introduce fields as the correct way to proceed? Here are several fantastic reasons:

- We already know classical fields are important in describing the physics of our world. For example, the theory of electromagnetism is governed by the electric/magnetic fields, which take values over all of space and time. So it seems natural to develop a quantum theory of fields if we want to quantize such theories (this motivation is more phenomenological and unlike the following ones, which proceed simply from trying to make the principles of quantum mechanics consistent with those of special relativity).
- Relativity says that rest mass is not a conserved quantity; instead, 4-momentum  $p^\mu$  is conserved, and  $p^0 = \gamma m$  for a particle of mass  $m$  moving at speed  $v$  with gamma factor  $\gamma = (1 - v^2/c^2)^{-1/2}$ . So, if rest mass is not conserved, there is no explicit conservation law forbidding, say, a particle of mass  $M$  decaying into two particles of mass  $m_1$  and  $m_2$  with  $m_1 + m_2 < M$ , with the difference in energy made up by their velocities. In physics, what is not expressly forbidden must occur. So we must be able

to accommodate it, and the non-relativistic Schrödinger equation doesn't. We will see that quantum fields allow this to occur. (A more phenomenological perspective is that we know atoms can have their electrons drop to a lower orbital and emit a photon, so we should accommodate particle creation).

- In the Klein-Gordon equation and the Dirac equation, we still have the structure familiar from the Schrödinger equation where there is a position operator  $\hat{x}$  but no time operator  $\hat{t}$ . Since special relativity unifies space and time we need to fix this. We can promote  $\hat{t}$  to an operator; this is the worldline formalism which we will not pursue. We will instead demote  $\hat{x}$  to just a position coordinate  $x$ . Both  $x$  and  $t$  will be arguments of quantum fields, e.g.  $\hat{\phi}(x, t)$ ; this operator is written in the Heisenberg picture where states are time-independent and operators are time-dependent,  $\hat{\phi}(x, t) = e^{i\hat{H}t/\hbar}\phi(x, 0)e^{-i\hat{H}t/\hbar}$ .<sup>1</sup> These objects  $\hat{\phi}$  and their canonical conjugates  $\hat{\pi}$  are what will enter into canonical commutation relations, instead of  $\hat{x}$  and  $\hat{p}$ . So, with all due apologies to position, its status must be demoted for us to continue with quantum field theory.

In physics, whenever a new framework is developed, it should reproduce the old framework in the appropriate limit. For example, special relativity reproduces Newtonian mechanics for  $v \ll c$  and general relativity reproduces Newtonian gravity when the field strengths are small. So whatever QFT we develop should reduce to nonrelativistic quantum-mechanics in an appropriate limit. It could be that the field concept is not useful in this limit, but it should still be there.<sup>2</sup> Rather than develop QFT and then take this limit (which would take a while to get to), let's just see if we can rewrite nonrelativistic quantum mechanics as a quantum theory of fields. We begin with the Schrödinger equation for  $n$  interacting particles, each of mass  $m$ :

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) = \left[ \sum_{j=1}^n \left( -\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j) \right) + \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \right] \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \quad (2.22)$$

Working in the Schrödinger picture we introduce the quantum field  $a(\mathbf{x})$  and  $a^\dagger(\mathbf{x})$  with commutation relations

$$[a(\mathbf{x}), a(\mathbf{x}')] = [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] = 0, \quad [a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}'). \quad (2.23)$$

<sup>1</sup>Recall that this is related to the Schrödinger picture where the states are time-dependent  $|\psi, t\rangle = e^{-iHt}|\psi\rangle$  by a change of basis by the unitary operator  $e^{-iHt}$ .

<sup>2</sup>A useful analogy is provided by general relativity; when field strengths are small we can just use Newtonian gravity, but if we took a small field-strength limit starting from general relativity we get a curved spacetime representation of Newtonian gravity.

These are fields because they depend on space (and in the Heisenberg picture would depend on time as well), and they are quantum because they obey a version of the commutation relations you are used to from quantum mechanics. They are like the ladder operators you are used to from the harmonic oscillator, except we have an independent set at every spatial point! We now propose the Hamiltonian

$$H = \int d^3x a^\dagger(\mathbf{x}) \left( -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) + \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}). \quad (2.24)$$

We define the vacuum state  $|0\rangle$  like in the harmonic oscillator:

$$a(\mathbf{x})|0\rangle = 0. \quad (2.25)$$

The vacuum is annihilated by the lowering operator at *any* spatial point  $\mathbf{x}$ . From this vacuum we can construct an excited state by applying the raising operators at arbitrary spatial points

$$|\psi, t\rangle = \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle. \quad (2.26)$$

With this definition of  $|\psi, t\rangle$  and Hamiltonian we can verify that the Schrödinger equation  $i\hbar\partial_t|\psi, t\rangle = H|\psi, t\rangle$  is satisfied if and only if  $\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t)$  satisfies (2.22).

Now we can interpret the creation operators as creating particles at particular positions. The vacuum  $|0\rangle$  is the state with no particles,  $a^\dagger(\mathbf{x}_1)|0\rangle$  is a state with a particle at position  $\mathbf{x}_1$ , etc. There is a conserved “number operator”

$$N = \int d^3x a^\dagger(\mathbf{x}) a(\mathbf{x}), \quad (2.27)$$

which counts the total number of particles. It is conserved because  $[N, H] = 0$ . (Notice for simplicity we are now dropping the hats on quantum operators, a convention we will mostly maintain in the rest of these notes – whether we have an operator or an eigenvalue should hopefully be clear from context, although it will take some getting used to!) So, as is appropriate for the non-relativistic Schrödinger equation, particles can move around and interact, but they cannot be created or destroyed.

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*Example 2:* Let’s check that  $N$  commutes with the potential term in  $H$ . We have the commutator

$$\int d^3z d^3x d^3y V(\mathbf{x} - \mathbf{y}) \left[ a^\dagger(\mathbf{z}) a(\mathbf{z}), a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \right] \quad (2.28)$$

where we pulled all c-numbers outside of the commutator. We can repeatedly apply the rule  $[AB, C] = A[B, C] + [A, C]B$  to simplify this expression, or we can write out the commutator expression and use  $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}') \implies a(\mathbf{x})a^\dagger(\mathbf{x}') = a^\dagger(\mathbf{x}')a(\mathbf{x}) + \delta^3(\mathbf{x} - \mathbf{x}')$  to move ladder operators past one another at the cost of a delta function (raising operators can move past other raising operators at no cost, similarly for lowering operators). So the first term of the commutator we can write as (ignoring the overall integration against  $V(\mathbf{x} - \mathbf{y})$ )

$$a^\dagger(\mathbf{z})a(\mathbf{z})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x}) = a^\dagger(\mathbf{z})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})a(\mathbf{z}) \quad (2.29)$$

$$+a^\dagger(\mathbf{z})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\delta^3(\mathbf{z} - \mathbf{x}) + a^\dagger(\mathbf{z})a^\dagger(\mathbf{x})a(\mathbf{y})a(\mathbf{x})\delta^3(\mathbf{z} - \mathbf{y}), \quad (2.30)$$

where we moved  $a(\mathbf{z})$  all the way to the right and picked up delta-function costs along the way. The second term of the commutator we can instead move  $a^\dagger(\mathbf{z})$  all the way to the left and pick up delta-function costs to get

$$a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{z})a(\mathbf{z}) = a^\dagger(\mathbf{z})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})a(\mathbf{z}) \quad (2.31)$$

$$+a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{z})\delta^3(\mathbf{x} - \mathbf{z}) + a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{x})a(\mathbf{z})\delta^3(\mathbf{y} - \mathbf{z}), \quad (2.32)$$

Using identities like  $\delta^3(\mathbf{x} - \mathbf{z}) = \delta^3(\mathbf{z} - \mathbf{x})$ ,  $a^\dagger(\mathbf{x})a^\dagger(\mathbf{y}) = a^\dagger(\mathbf{y})a^\dagger(\mathbf{x})$ , and  $a(\mathbf{z})\delta^3(\mathbf{y} - \mathbf{z}) = a(\mathbf{y})\delta^3(\mathbf{y} - \mathbf{z})$ , we see that the difference of these two terms vanishes.

The commutation of  $N$  with  $H$  was guaranteed by the fact that  $H$  had the same number of raising and lowering operators in each term in  $H$ . But now we can imagine changing this by adding a term to the Hamiltonian

$$\Delta H \propto \int d^3x \left[ a^\dagger(\mathbf{x})a^2(\mathbf{x}) + a^\dagger(\mathbf{x})^2a(\mathbf{x}) \right]. \quad (2.33)$$

Looking back at the exercise above, we see that when we move  $a$  to the right of  $\Delta H$  in the first term of the commutator we will pick up one delta function, but when we move  $a^\dagger$  to the left we will pick up two delta functions. So their difference (even after doing the spatial integrals) will not vanish.

*Example 3:* We claimed that the operator

$$N = \int d^3x a^\dagger(\mathbf{x})a(\mathbf{x}) \quad (2.34)$$

which counts the total number of particles is conserved for our proposed Hamiltonian (2.24).

But we noticed something even stronger is true, at least for the potential term in the Hamiltonian. In that case we didn't even need to do the spatial integrals to see that the operators commuted: the integrand of the potential term commuted with the integrand of  $N$ . This means the potential term conserves the number of particles at any given spatial point. This is because each  $a(\mathbf{x})$  comes with an  $a^\dagger(\mathbf{x})$  *at the same spatial point*. So if you annihilate a particle at a spatial point you also have to create one at the same point. The kinetic term in  $H$  violates this due to the gradient term; if you repeat the exercise to compute the commutator you will find a derivative of a delta function that appears for one term in the commutator and not the other. This term vanishes only when you integrate over all of space. Intuitively, this is because the kinetic term is responsible for propagation: you annihilate a particle at  $\mathbf{x}$ , propagate a little bit with  $\nabla^2$ , and then create a particle nearby. This intuition comes from Taylor expanding, e.g. for a function of a scalar variable  $a(x + \epsilon) = a(x) + \epsilon a'(x) + \epsilon^2 a''(x)/2 + \dots$ . So the second derivative is constructing the function nearby, at  $x + \epsilon$ .

We can write a potential term which also annihilates a particle in one place and creates it someplace else. Then we should again see that such a term keeps the total number of particles conserved, but not the number at a given spatial point. We write

$$\Delta H \propto \int d^3x d^3y a^\dagger(\mathbf{x})a(\mathbf{y}) \quad (2.35)$$

We don't add the Hermitian conjugate because this is already Hermitian, like the terms in (2.24). But it is nonlocal, since it is written as a double integral and has the physical effect we mentioned before. We have

$$[N, \Delta H] = \int d^3z d^3x d^3y \left( a^\dagger(\mathbf{z})a(\mathbf{y})\delta^3(\mathbf{z} - \mathbf{x}) - a^\dagger(\mathbf{x})a(\mathbf{z})\delta^3(\mathbf{y} - \mathbf{z}) \right) \quad (2.36)$$

The integrand is nonzero (since the delta functions are different), reflecting the fact that this term breaks conservation of particle number at a spatial point, but the integral vanishes, reflecting the fact that the total number of particles is conserved.

---

Let's return to (2.26) and define

$$\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) \equiv M_{\mathbf{x}_1 \dots \mathbf{x}_n} X_{\mathbf{x}_1 \dots \mathbf{x}_n} \quad (2.37)$$

By the commutativity of the creation operators we have that  $X$  is totally symmetric in all its indices  $\mathbf{x}_1$  through  $\mathbf{x}_n$ . This means that we can restrict to tensors  $M$  that are totally symmetric in all their indices, since any antisymmetric piece will vanish after integration

in the expression (2.26).<sup>3</sup> Such wavefunctions have Bose-Einstein statistics and describe bosons. For Fermi-Dirac statistics, we have to define *anticommutation* relations for our quantum fields:

$$\{a(\mathbf{x}), a(\mathbf{x}')\} = \{a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')\} = 0, \quad \{a(\mathbf{x}), a^\dagger(\mathbf{x}')\} = \delta^3(\mathbf{x} - \mathbf{x}') \quad (2.38)$$

where  $\{A, B\} \equiv AB + BA$  is called the anticommutator. In this case, we have that only the totally antisymmetric part of  $\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t)$  survives the integration in (2.26). This is a theory of fermions.<sup>4</sup>

We now turn to developing a relativistic QFT. We will see that it will solve all of our problems very elegantly. It will allow for particle creation/annihilation, and it will resolve the acausal propagation we explored in Example 1. It will do the latter by requiring the existence of antiparticles. Whenever we see a nonzero value for a propagator from  $\mathbf{x}_1$  to  $\mathbf{x}_2$ , it will be cancelled by an equal and opposite contribution by an antiparticle propagator from  $\mathbf{x}_2$  to  $\mathbf{x}_1$ !

Before beginning, we will review the harmonic oscillator and classical field theory in the next two sections, since (perturbative) QFT is to some extent a merger of these two concepts.

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<sup>3</sup>Recall that any two-dimensional matrix  $M_{ij}$  can be written as the sum of a symmetric matrix and antisymmetric matrix,  $M_{ij} = S_{ij} + A_{ij}$  with  $2S_{ij} = M_{ij} + M_{ji}$ ,  $2A_{ij} = M_{ij} - M_{ji}$ , and that a contraction of a symmetric matrix  $S_{ij}$  with an antisymmetric matrix  $A_{ij}$  gives  $S_{ij}A^{ij} = -S_{ji}A^{ji} = -S_{ij}A^{ij} \implies S_{ij}A^{ij} = 0$ . Higher-rank tensors can be written as the sum of a totally symmetric tensor, a totally antisymmetric tensor, and a mixed symmetry tensor. But the relevant point is that the mixed symmetry tensor will be antisymmetric in some pair of indices, which will lead to the vanishing of (2.26).

<sup>4</sup>When we write a relativistic QFT, it will have to obey the *spin-statistics* theorem, which says that interacting bosons have integer spin while interacting fermions have half-integer spin. In our example above the particles have spin zero (since there is no label on the quantum fields that could account for spin), yet they could be bosons or fermions depending on whether we defined commutation relations or anticommutation relations. See the beginning of Section 3.5 of [5] for an illustration of what goes wrong when you try to give fermionic quantum fields commutation relations. Spoiler: the energy becomes unbounded below.

### 3 The harmonic oscillator

In the words of Sidney Coleman, "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction." So let's go through a few levels now. The classical equation of motion is

$$m \frac{d^2 q}{dt^2} + kq = 0. \quad (3.1)$$

This has plane wave solutions  $e^{\pm i\omega t}$  with  $\omega = +\sqrt{k/m}$ . The general solution is therefore a linear combination

$$q(t) = a_1 e^{-i\omega t} + a_2 e^{i\omega t}, \quad \omega = \sqrt{k/m}. \quad (3.2)$$

Requiring  $q$  be real,  $q^* = q$ , enforces  $a_2 = a_1^* := a^*$ , which gives

$$q(t) = a e^{-i\omega t} + a^* e^{i\omega t}, \quad \omega = \sqrt{k/m}. \quad (3.3)$$

The classical Hamiltonian is given by kinetic plus potential energy:

$$H(q, p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \quad (3.4)$$

We can obtain the Lagrangian from this by a Legendre transform

$$L(q, \dot{q}) = p(q, \dot{q})\dot{q} - H(q, p(q, \dot{q})), \quad \dot{q} := \frac{\partial H(q, p)}{\partial p} \quad (3.5)$$

Notice on the RHS of the first expression the momentum  $p$  is written in terms of  $q, \dot{q}$  since the Lagrangian is solely in terms of  $q, \dot{q}$ . We solve for  $p(q, \dot{q})$  using the definition of  $\dot{q}$  to get  $p(q, \dot{q}) = m\dot{q}$  and plug into the expression for the Lagrangian to obtain

$$L(q, \dot{q}) = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2}. \quad (3.6)$$

This is just the kinetic energy minus the potential energy. We can Legendre transform back to the Hamiltonian using

$$H(q, p) = p\dot{q}(q, p) - L(q, \dot{q}(q, p)), \quad p := \frac{\partial L(q, \dot{q})}{\partial \dot{q}}. \quad (3.7)$$

Again, we solve the latter equation to get  $\dot{q}(q, p) = p/m$  and plug into the former equation. The equation of motion (3.1) is obtained by the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \implies -m\omega^2 q - \frac{d}{dt} m\dot{q} = 0 \implies (3.1). \quad (3.8)$$

**Given a classical Lagrangian, we can quantize the theory by computing the momentum  $p$  conjugate to the variable  $q$  and then imposing a commutation relation:**

$$[q, p] = i\hbar. \quad (3.9)$$

**we will do precisely this when we get to QFT.** You may recall that, after the introduction of ladder operators  $a, a^\dagger$  with

$$[a, a^\dagger] = 1, \quad (3.10)$$

we have the representation

$$q = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \quad H = \hbar\omega(a^\dagger a + \frac{1}{2}) \quad (3.11)$$

The time dependence of an operator  $O(t)$ , in the Heisenberg picture, is given by

$$O(t) = e^{iHt}O(0)e^{-iHt} \implies i\hbar \frac{d}{dt}O(t) = [O, H]. \quad (3.12)$$

This means our ladder operator satisfy

$$i\hbar \frac{d}{dt}a = [a, H] = [a, \hbar\omega(a^\dagger a + \frac{1}{2})] = \hbar\omega a, \quad i\hbar \frac{d}{dt}a^\dagger = [a^\dagger, H] = -\hbar\omega a^\dagger. \quad (3.13)$$

The second equation is the complex conjugate of the first, as required. These operator equations have solutions

$$a(t) = e^{-i\omega t}a(0), \quad a(t)^\dagger = e^{i\omega t}a(0)^\dagger. \quad (3.14)$$

Recalling our representation of  $q$  in terms of ladder operators, we can now write

$$q(t) = \sqrt{\frac{\hbar}{2m\omega}} \left( a e^{-i\omega t} + a^\dagger e^{i\omega t} \right). \quad (3.15)$$

Notice the similarity to (3.3)! The coefficients  $a, a^*$  have become the quantum ladder operators  $a, a^\dagger$  (so the complex conjugate got promoted to a Hermitian conjugate), with a trivial

change in normalization leading to the  $\sqrt{\hbar/(2m\omega)}$  prefactor. We will see the same thing occur when we go from classical field theory to quantum field theory below. This representation of the Heisenberg picture operator  $q(t)$  is not often shown in introductory quantum mechanics since they almost exclusively work in the Schrödinger picture where you have a time-independent operator  $q$ .

## 4 Classical field theory

Before developing quantum field theory we should review a bit of classical field theory. This describes a classical system where the fundamental variables describing the system are fields, i.e. continuous functions of space and time. The electric and magnetic fields are a good example. We will focus on classical field theories with Lorentz invariance. They can be studied either through their Hamiltonian or their Lagrangian. We will introduce both but will focus on the Lagrangian description when we develop QFT. I strongly suggest reviewing Appendix A for some mathematical details, in particular about somewhat precise definitions regarding functionals and variations.

### 4.1 Hamiltonians, Lagrangians, and the Euler-Lagrange equations

The Hamiltonian and Lagrangian can be written as integrals over space of a Hamiltonian density  $\mathcal{H}$  or Lagrangian density  $\mathcal{L}$ , respectively.<sup>5</sup>

$$H[\phi, \pi] = \int d^3x \mathcal{H}(\phi, \partial_i \phi, \pi), \quad L[\phi, \dot{\phi}] = \int d^3x \mathcal{L}(\phi, \partial_i \phi, \dot{\phi}). \quad (4.1)$$

Usually we also call the unintegrated quantities  $\mathcal{H}$  and  $\mathcal{L}$  the Hamiltonian and Lagrangian. Like in classical mechanics (cf (3.5) and (3.7)), the Hamiltonian and Lagrangian can be obtained as Legendre transforms of one another:

$$\mathcal{L}(\phi, \partial_i \phi, \dot{\phi}) = \pi(\phi, \partial_i \phi, \dot{\phi}) \dot{\phi} - \mathcal{H}(\phi, \partial_i \phi, \pi(\phi, \partial_i \phi, \dot{\phi})), \quad \dot{\phi} := \frac{\partial \mathcal{H}(\phi, \partial_i \phi, \pi)}{\partial \pi}. \quad (4.2)$$

Time derivatives are defined as  $\dot{\phi} = \partial_t \phi = -\partial^t \phi$ .

Practically, to obtain the Lagrangian from the Hamiltonian, we solve the second equation for  $\pi$  as a function of  $\phi$ ,  $\partial_i \phi$ , and  $\dot{\phi}$  and plug into the first equation. The inverse transform is defined by

$$\mathcal{H}(\phi, \partial_i \phi, \pi) = \pi \dot{\phi}(\phi, \partial_i \phi, \pi) - \mathcal{L}(\phi, \partial_i \phi, \dot{\phi}(\phi, \partial_i \phi, \pi)), \quad \pi := \frac{\partial \mathcal{L}(\phi, \partial_i \phi, \dot{\phi})}{\partial \dot{\phi}}. \quad (4.3)$$

To obtain the Hamiltonian from the Lagrangian, we solve the second equation for  $\dot{\phi}$  in terms of  $\phi$ ,  $\partial_i \phi$ , and  $\pi$  and plug into the first equation. From here on out we will not be as careful with the arguments of our functions!

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<sup>5</sup>As discussed in Appendix A, the Lagrangian density and Hamiltonian density are functions, not functionals, and this is represented with round brackets. So we leave off the spacetime dependence of the fields to not clutter the notation.

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*Example 1:* Let's compute the Hamiltonian from the Lagrangian in a simple example. The Lagrangian for a scalar field  $\phi$  with potential  $V(\phi)$  looks like:<sup>6</sup>

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi). \quad (4.4)$$

The canonically conjugate momentum is given by  $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$ , which gives  $\dot{\phi}(\phi, \partial_i\phi, \pi) = \pi$ . This means the Hamiltonian is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi). \quad (4.5)$$

We see that we can again interpret the Hamiltonian as a sum of kinetic ( $\frac{1}{2}\pi^2$ ) and potential ( $\frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$ ) energy, while the Lagrangian is the kinetic ( $\frac{1}{2}\dot{\phi}^2$ ) minus the potential ( $\frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$ ) energy

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We will soon see that the Hamiltonian  $H = \int d^3x\mathcal{H}$  corresponds to a conserved quantity, the total energy, while the Lagrangian does not. While this is nice, it also means the description of the system is not manifestly Lorentz invariant, since the energy is the 0 component of the four-momentum  $p^\mu = (E, p^i)$ . The energy density  $\mathcal{H}$  is instead the 00 component of the energy-momentum tensor  $T^{\mu\nu}$ . In fact we can see from our example above that  $\mathcal{H}$  is not a Lorentz scalar. For this reason we will focus on the Lagrangian, which is manifestly Lorentz invariant since it is a Lorentz scalar.

Using the Lagrangian we can define the action as usual,

$$S[\phi] = \int dt L[\phi(t), \dot{\phi}(t)] = \int d^4x \mathcal{L}(\phi(x), \partial_\mu\phi(x)). \quad (4.6)$$

We get the equations of motion by imposing the principle of least action,  $\delta S = 0$ . To vary the action we imagine varying  $\phi \rightarrow \phi + \alpha\delta\phi$  for infinitesimal  $\alpha$  and arbitrary  $\delta\phi$  *which vanishes at any boundaries of spacetime*. Restricting to Lagrangians that are just a function of the field and its first derivative,  $\mathcal{L}(\phi, \partial_\mu\phi)$ , we get

$$\delta S[\phi] = \int d^4x \left[ \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta(\partial_\mu\phi) \right] \quad (4.7)$$

---

<sup>6</sup>This is the first place where you will see our use of  $(-, +, +, +)$  signature give a formula that looks different than what you'd get with  $(+, -, -, -)$  signature; in the latter case the derivative term is  $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ . When expanded into components, either signature gives a positive time-derivative term and negative spatial derivatives.

The variation of  $\phi$  everywhere in spacetime has induced variations of  $\phi(x)$  and  $\partial_\mu\phi(x)$ , i.e. the field and its first derivative at a given spacetime point.

Using (A.7) to write  $\delta(\partial_\mu\phi(x)) = \partial_\mu(\delta\phi(x))$  and integrating by parts (ignoring the boundary term since we assumed the variation  $\delta\phi$  vanishes there), we get

$$\delta S[\phi] = \int d^4x \left( \frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} \right) \delta\phi(x). \quad (4.8)$$

We demand stationarity of the action  $\delta S = 0$  for *arbitrary* variation  $\delta\phi$ , which gives

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \quad (4.9)$$

by the fundamental lemma of the calculus of variations.

---

*Example 2:* Let's compute the Euler-Lagrange equations for a self-interacting scalar field theory:

$$S = \int d^4x \left( -\frac{1}{2} \partial_\mu\phi\partial^\mu\phi - V(\phi) \right). \quad (4.10)$$

We have

$$\frac{\partial\mathcal{L}}{\partial\phi} = -V'(\phi), \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -\frac{1}{2} \frac{\partial(\partial_\nu\phi\partial^\nu\phi)}{\partial(\partial_\mu\phi)} = -\frac{1}{2} (\delta^\mu_\nu\partial^\nu\phi + \delta^{\mu\nu}\partial_\nu\phi) = -\partial^\mu\phi. \quad (4.11)$$

$$\implies \partial_\mu\partial^\mu\phi = V'(\phi) \implies \square\phi - V'(\phi) = 0, \quad (4.12)$$

where we defined  $\square = \partial_\mu\partial^\mu = \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$ , the d'Alembert operator or d'Alembertian. So in components we have  $\square = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$  (due to our metric signature  $(-, +, +, +)$ ), which has an overall minus sign to how the operator is defined in some references (so in those references the EOM would be written  $\square\phi + V'(\phi) = 0$ ).

If we pick  $V(\phi) = m^2\phi^2/2$ , then we get the Klein-Gordon equation:

$$(\square - m^2)\phi = 0. \quad (4.13)$$

This is the equation of motion for a massive, free scalar field.

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## 4.2 Noether's theorem

The theorem due to Emmy Noether has deep impact throughout many branches of physics. It says that any system with a Lagrangian description that is invariant under a continuous symmetry has a locally conserved quantity. **A symmetry is a transformation of the fields that leaves the Lagrangian invariant up to a total divergence, and therefore leaves the equations of motion invariant.**<sup>7</sup> Some examples of Noether's theorem: translation invariance in space implies momentum conservation and translation invariance in time implies energy conservation. Rather than investigating what field transformations do to the EOM directly, we will instead investigate what they do to the Lagrangian (and therefore the action, which is the fundamental object).

The massive scalar field Lagrangian  $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - m^2\phi^2/2$  has a *discrete* symmetry,  $\phi \rightarrow -\phi$ . It is discrete because you cannot do the transformation by an arbitrary "small" amount: you either do it or you don't. To get a continuous symmetry lets consider a *complex* scalar field  $\phi$  with mass  $m$ , which has Lagrangian

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2. \quad (4.14)$$

This is still invariant under  $\phi \rightarrow -\phi$ , but notice there is a bigger symmetry group,  $\phi \rightarrow e^{-i\alpha}\phi$  for any  $\alpha \in \mathbb{R}$ . This is a complex phase rotation of  $\phi$ , by an *arbitrary* amount  $\alpha$ . So we can do the transformation by a small amount (the case  $\alpha = \pi$  gives us the transformation  $\phi \rightarrow -\phi$ ). A nice way to think about the complex scalar field is that it is two real scalar fields,  $\phi = \phi_1 + i\phi_2$ , or as two complex conjugate fields  $\phi, \phi^* = \phi_1 - i\phi_2$ . The Lagrangian can be written in the latter basis as

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*, \quad (4.15)$$

and the symmetry transformation is

$$\phi \rightarrow e^{-i\alpha}\phi, \quad \phi^* \rightarrow e^{i\alpha}\phi^*. \quad (4.16)$$

**Unlike when we derived the Euler-Lagrange equations, in this case  $\delta\phi_i$  need not go to zero at the boundaries of spacetime.** Let's now prove Noether's theorem.

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<sup>7</sup>There also exist symmetries of the equations of motion that do not transform the Lagrangian by a total derivative, e.g. electromagnetic duality in Maxwell theory, the discrete version of which is swapping the electric and magnetic fields. This changes the Lagrangian  $\mathbf{E}^2 - \mathbf{B}^2$  by flipping the overall sign. Continuous symmetries of the equations of motion which do not leave the Lagrangian invariant up to a total divergence do not have corresponding conserved quantities derivable from the procedure in this section.

**Warm-up Noether's theorem:** Consider a Lagrangian density that is a function of  $N$  scalar fields  $\phi_i$ , and a symmetry transformation that can mix any subset of these  $N$  scalar fields,  $\phi_i \rightarrow \phi_i + \alpha \delta\phi_i$  with infinitesimal  $\alpha$ . We assume that the symmetry leaves the Lagrangian invariant,  $\delta\mathcal{L} = 0$ . Then there exists a conserved current  $j^\mu$ :

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i. \quad (4.17)$$

**Proof:** The change in the Lagrangian density, as shown in (4.7), is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i} \delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta(\partial_\mu\phi_i). \quad (4.18)$$

Since the  $i$  index is contracted there is an implicit sum over the  $N$  fields. We use  $\delta(\partial_\mu\phi_i) = \partial_\mu(\delta\phi_i)$  from (A.7) to rewrite this as

$$\delta\mathcal{L} = \left( \frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \right) \delta\phi_i + \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i \right). \quad (4.19)$$

In this case we do not drop the total derivative, since  $\delta\phi_i$  and  $\partial\mathcal{L}/\partial(\partial_\mu\phi_i)$  do not have to go to zero at the boundaries, so they can contribute to  $\delta S$ . This is a totally general expression for the variation of a Lagrangian with dependence  $\mathcal{L}(\phi, \partial_\mu\phi)$ .

To derive the Noether current, we will now impose the EOM. This means the current will only be conserved when the EOM are satisfied.<sup>8</sup> We get

$$\delta\mathcal{L} = \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i \right). \quad (4.20)$$

Since this equals zero by assumption, we get (4.17).

The above is the form of Noether's theorem that we will mostly need. But there is a more powerful version of the theorem. The basic point is that  $\delta\mathcal{L} = 0$  was too restrictive – the Lagrangian density can change by a boundary term, and this won't affect the equations of motion, since the equations of motion are derived by arbitrary variations *which don't mess with the boundary, i.e. they go to zero there*. So let's prove the more powerful theorem.

**General Noether's theorem:** Consider a Lagrangian density with symmetry transformation as before,  $\phi_i \rightarrow \phi_i + \alpha \delta\phi_i$  with infinitesimal  $\alpha$ . We assume that the symmetry leaves

<sup>8</sup>There are more powerful currents that are conserved even when the EOM are not satisfied, for example in 2d scalar field theory we have  $j^\mu = \varepsilon^{\mu\nu} \partial_\nu\phi$ .

the Lagrangian invariant up to a total divergence,

$$\delta\mathcal{L} = \partial_\mu K^\mu. \quad (4.21)$$

Then there exists a conserved current  $j^\mu$ :

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i - K^\mu. \quad (4.22)$$

**Proof:** We have the same expression (4.20) for the variation of the Lagrangian density once the EOM are imposed, except now we have to set it equal to (4.21). This gives

$$\partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i \right) = \partial_\mu K^\mu, \quad (4.23)$$

which we can rewrite as (4.22).

From a conserved current  $j^\mu$  one can always define a conserved charge as

$$Q = \int d^3x j^0, \quad \frac{d}{dt}Q = 0. \quad (4.24)$$

Now that we have proved the theorems, we would like to apply them in certain examples. The general Noether theorem looks a bit perplexing from this point of view: to get a concrete expression for  $j^\mu$ , we need two expressions for  $\delta\mathcal{L}$ , since they are equated in the proof of the theorem. But if we use literally the same expression for both, then we will find  $j^\mu = 0$ . This is conserved but useless. The basic point is that in any example we want to see that (4.21) is true *without* using the equations of motion.<sup>9</sup> This is because (4.21) is our *statement* of what a symmetry is, and a symmetry leaves our EOM unchanged. We should not use the EOM to show that the EOM remain unchanged!

*Example 3:* Let's compute the Noether current coming from the symmetry  $\phi \rightarrow e^{-i\alpha}\phi$ ,  $\phi^* \rightarrow e^{i\alpha}\phi^*$  acting on the Lagrangian  $\mathcal{L} = \partial_\nu\phi\partial^\nu\phi^* - m^2\phi\phi^*$ . The Lagrangian is invariant under this transformation,  $\delta\mathcal{L} = 0$ , so  $K^\mu$  is a constant vector which we can take to be zero, since the conservation of a constant vector is not terribly exciting. Using

$$\delta\phi := \left. \frac{\partial\phi}{\partial\alpha} \right|_{\alpha=0} = -i\phi, \quad \delta\phi^* = i\phi^* \quad (4.25)$$

<sup>9</sup>Using the EOM one will just find (4.20), i.e.  $K^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i$ , leading to  $j^\mu = 0$ . But if we don't impose the EOM, then once we strip the derivative from  $\partial_\mu K^\mu$  to define  $K^\mu$ , imposing the EOM will no longer lead to  $K^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i$

gives

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta \phi^* = -i\phi \delta_\nu^\mu \partial^\nu \phi^* + i\phi^* \delta^{\nu\mu} \partial_\nu \phi = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*). \quad (4.26)$$

Notice the current is real, as expected since the Lagrangian is real.

*Example 4:* Let's consider an example with  $K^\mu \neq 0$ . This example will be a *spacetime* symmetry, where the transformation of the fields is induced by a transformation of the spacetime point they depend on. We consider spacetime translations

$$x^\mu \rightarrow x^\mu - \alpha \xi^\mu \quad (4.27)$$

for some (arbitrary) constant  $\xi^\mu$ . The Lagrangian is a scalar function of the spacetime position, so Taylor expanding around  $\alpha = 0$  to first order gives

$$\mathcal{L}(x + \alpha \xi) = \mathcal{L}(x) + \alpha \xi^\nu \partial_\nu \mathcal{L}(x) + \mathcal{O}(\alpha^2) \implies \delta \mathcal{L} = \partial_\nu (\xi^\nu \mathcal{L}). \quad (4.28)$$

Similarly, the transformation of a field gives<sup>10</sup>

$$\phi(x) \rightarrow \phi_i(x + \alpha \xi) = \phi(x) + \xi^\nu \partial_\nu \phi(x) + \mathcal{O}(\alpha^2) \implies \delta \phi = \xi^\nu \partial_\nu \phi. \quad (4.29)$$

Phrased this way, we interpret the symmetry transformation as replacing the field by itself plus some derivative of itself, *at the same spacetime point*. So the coordinates don't change. **This is the right way to think since the fundamental object is the action, which just depends on field configurations over all spacetime, not coordinates.**

We compute the Noether current

$$j^\mu = \xi^\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \xi^\mu \mathcal{L} \quad (4.30)$$

This gives four independent currents, one for each choice of direction to translate in, i.e. the choices  $\xi^\mu \in \{\delta_t^\mu, \delta_x^\mu, \delta_y^\mu, \delta_z^\mu\}$ . To see this, we can repackage the conserved currents into an

<sup>10</sup>To understand why the coordinate change  $x \rightarrow x - \alpha \xi$  corresponds to transforming a field as  $\phi(x + \alpha \xi)$ , it helps to picture a field  $\phi$  depending on two spatial coordinates drawn on a piece of paper. Place this on top of another piece of paper that has the coordinates  $x$  drawn. Transforming the coordinates (the bottom piece of paper) one way is the same as transforming the field (the top piece of paper) the opposite way.

object known as the stress tensor  $T_\nu^\mu$  through  $j^\mu = -\xi^\nu T_\nu^\mu$ , which gives

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i + \eta^{\mu\nu} \mathcal{L}. \quad (4.31)$$

Another way to see this is to “factor out”  $\xi^\nu$  from (4.30). Since the term  $\xi^\mu \mathcal{L}$  has a  $\mu$  index instead of a  $\nu$  index we need to stick in a  $\eta_\nu^\mu$  to change the index, and that gives us  $T_\nu^\mu$  and therefore  $T^{\mu\nu}$ .

This seems too good to be true, since almost no assumptions went into this derivation. Indeed, the stress tensor gives a set of four conserved currents for any Lorentz-invariant quantum field theory. The important (implicit) assumption was that the Lagrangian density did not depend on spacetime position explicitly; if it did, then recalling that we transform the fields but *not* the coordinates means that  $\mathcal{L}$  does not transform as a total derivative, since we cannot obtain its transformation law by  $\mathcal{L}(x + a\xi) = \mathcal{L}(x) + \alpha\delta\mathcal{L}(x)$ , as that would transform the explicit coordinates as well.

For a scalar field theory  $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$  we have (recall  $\dot{\phi} = \partial_t\phi = -\partial^t\phi$ )

$$T^{00} = \dot{\phi}^2 - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \quad (4.32)$$

which agrees with the energy density  $\mathcal{H}$  obtained in (4.5) by Legendre transforming the Lagrangian. The momentum densities are given by

$$T^{0j} = -\dot{\phi}\partial^j\phi = -\pi\partial^j\phi. \quad (4.33)$$

So we have the energy-momentum four-vector

$$P^\mu = \int d^3x T^{0\mu}. \quad (4.34)$$

The other diagonal components  $T^{jj}$  in the stress energy tensor have the interpretation of pressures, while the off-diagonal components  $T^{jk}$  have the interpretation of shears.

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## 5 Canonical quantization of scalar fields

We consider free, massive scalar field theory:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0. \quad (5.1)$$

We have allowed ourselves an arbitrary additive constant  $\Omega_0$ , which does not affect the Euler-Lagrange equations:

$$\delta S = 0 \implies (-\partial^2 + m^2)\phi = 0. \quad (5.2)$$

Plane waves  $e^{i\mathbf{k}\cdot\mathbf{x}\pm i\omega_{\mathbf{k}}t}$  are solutions to this equation, with arbitrary momentum  $\mathbf{k}$  and frequency satisfying  $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$ . We take  $\omega_{\mathbf{k}} \geq 0$  and write the general solution as an arbitrary superposition of such plane waves

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{f(k)} \left[ a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} + b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t} \right]. \quad (5.3)$$

The function  $f(k)$  can be absorbed into the  $a(\mathbf{k})$  and  $b(\mathbf{k})$ , but we will keep it for reasons soon to be apparent.<sup>11</sup> We need to impose  $\phi^* = \phi$ , since our scalar field is real. This gives

$$\phi^*(\mathbf{x}, t) = \int \frac{d^3k}{f(k)} \left[ a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t} + b^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} \right] \quad (5.4)$$

$$= \int \frac{d^3k}{f(k)} \left[ a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t} + b^*(-\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} \right] \quad (5.5)$$

where we relabeled the dummy variable  $\mathbf{k} \rightarrow -\mathbf{k}$  in the  $b^*(\mathbf{k})$  term. Equating with (5.3) requires  $b^*(-\mathbf{k}) = a(\mathbf{k})$ , which gives

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{f(k)} \left[ a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} + a^*(-\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t} \right] \quad (5.6)$$

$$= \int \frac{d^3k}{f(k)} \left[ a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} + a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t} \right] \quad (5.7)$$

$$= \int \frac{d^3k}{f(k)} \left[ a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx} \right]. \quad (5.8)$$

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<sup>11</sup>Recall that plane-wave solutions to the nonrelativistic Schrödinger equation look like  $\psi \sim e^{i\mathbf{k}\cdot\mathbf{x}-iE(\mathbf{k})t}$  with  $E(\mathbf{k}) = \mathbf{k}^2/(2m)$ . This is why we cannot interpret the field  $\phi$  as a wavefunction, since the  $b(\mathbf{k})$  term would correspond to negative-energy excitations  $E(\mathbf{k}) = -\omega_{\mathbf{k}} < 0$ .

We defined the shorthand  $kx := k^\mu x_\mu = \eta_{\mu\nu} k^\mu x^\nu = -\omega_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x}$  where  $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$  and  $x^\mu = (t, \mathbf{x})$ . So we have  $k^2 = \mathbf{k}^2 - k_0^2$  and to satisfy the Euler-Lagrange equations we have

$$k^2 = -m^2 \implies k_0^2 = \omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2. \quad (5.9)$$

A four-momentum which satisfies this is called “on the mass shell” or “on shell.” This just means the EOM are satisfied.

We will now use our freedom in  $f(k)$  to fix a convention. We would like to interpret  $d^3k/f(k)$  as an integration measure, and part of our convention will be that it should be Lorentz-invariant. This fixes  $f(k) \sim \omega_{\mathbf{k}}$ . This can be seen by beginning with a Lorentz invariant measure

$$\frac{d^4k}{(2\pi)^3} \delta(k^2 + m^2) \Theta(k^0), \quad (5.10)$$

where  $\Theta$  is the Heaviside step function and the  $(2\pi)^{-3}$  is a convention. The  $d^4k$  is Lorentz invariant since a Lorentz transformation  $k^\mu \rightarrow \Lambda^\mu{}_\nu k^\nu$  gives  $d^4k = |\det \Lambda| d^4k = d^4k$ . The delta function is Lorentz invariant since the argument is a Lorentz scalar with all indices contracted. The Heaviside step function looks non-invariant since it singles out  $k^0$ , but notice that it equals 1 for all  $k^0 \geq 0$  and equals 0 otherwise. The sign of the zero component of a 4-vector cannot be changed under a Lorentz transformation, so this is also invariant. In this expression we think of  $k^\mu = (k^0, k^i)$  with  $k^0$  general, i.e. not necessarily equal to  $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$  (although the point of the delta function will be to enforce this). Integrating over the measure gives

$$\int d^4k \delta(k^2 + m^2) \Theta(k^0) F(k^0, \mathbf{k}) = \int d^3k \int_{-\infty}^{\infty} dk^0 \delta(k^2 + m^2) \Theta(k^0) F(k^0, \mathbf{k}) = \frac{1}{2\omega_{\mathbf{k}}} F(\omega_{\mathbf{k}}, \mathbf{k}) \quad (5.11)$$

where we used

$$\int_{-\infty}^{\infty} dx \delta(g(x)) F(x) = \sum_i \frac{F(x_i)}{|g'(x_i)|} \quad (5.12)$$

where the  $x_i$  are simple zeroes of  $g(x)$ .  $F(k^0, \mathbf{k})$  is like the term in brackets in (5.8), but with  $k^0$  general, i.e. not equal to  $\omega_{\mathbf{k}}$ . The point of the Heaviside function was to eliminate the zero  $k^0 = -\omega_{\mathbf{k}}$ , so that we get (5.8) once the delta function fixes  $k^0 = \omega_{\mathbf{k}}$ . So we finally have

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[ a(\mathbf{k}) e^{ikx} + a^*(\mathbf{k}) e^{-ikx} \right] \quad (5.13)$$

Try to work out the inversion formula which gives  $a(\mathbf{k})$  in terms of  $\phi(x)$ ; if stuck, consult Chapter 3 of [1].

**Recall that we wrote the classical solution to the harmonic oscillator as  $q = ae^{-i\omega t} + a^*e^{i\omega t}$  in (3.3). When passing to the quantum harmonic oscillator  $a, a^*$  became the ladder operators  $a, a^\dagger$  in the expression for the position operator  $\hat{q}$  in (3.15). Here we have an infinite family of harmonic oscillators, one for each  $\mathbf{k}$ , so we should expect the same to occur when passing from the classical field  $\phi(x)$  to the quantum field operator  $\hat{\phi}(x)$ , and indeed that is precisely what will happen!**

To quantize the theory, we first obtain the Hamiltonian density  $\mathcal{H}(\phi, \pi)$ . This was done in (4.5):

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Omega_0. \quad (5.14)$$

We can write a classical expression for the Hamiltonian  $H = \int d^3x \mathcal{H}$  using (5.13) and  $\pi(x) = \dot{\phi}(x)$  (see (3.26) of [1] for details):

$$-\Omega_0 V + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \omega_{\mathbf{k}} (a^*(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^*(\mathbf{k})), \quad (5.15)$$

where we have not commuted  $a$  and  $a^*$  past one another in anticipation of going to the quantum theory.

Now let's pass to the quantum theory by imposing canonical commutation relations. These are done on the variables/fields of the theory and their canonically conjugate momenta. For example in multiparticle quantum mechanics we impose

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\delta_{ij}. \quad (5.16)$$

These are usually thought of as commutators in the Schrödinger picture, but they can be interpreted as equal-time commutators in the Heisenberg picture, i.e.  $[q_i(t), p_j(t)] = i$ . In our case, we want to take this finite discrete set into a continuous infinity of spacetime points. Again doing this at equal times we impose

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0, \quad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}'). \quad (5.17)$$

These relations imply

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0, \quad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (5.18)$$

While the universal approach to quantization is to impose the commutation relations on the dynamical fields and their conjugate momenta, it may help bridge the gap to ordinary quantum mechanics to imagine imposing them on the infinity of ladder operators (5.18) and seeing what this says about the fields and their momenta (5.17).

As predicted above, the quantum field becomes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left( a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx} \right), \quad (5.19)$$

where the classical fields  $a(\mathbf{k})$ ,  $a^*(\mathbf{k})$  have been upgraded to quantum operators  $a(\mathbf{k})$ ,  $a^\dagger(\mathbf{k})$ . This quantum field satisfies the Klein-Gordon equation  $(\partial^2 - m^2)\phi = 0$  just like the classical field. The quantum Hamiltonian  $H = \int d^3x \mathcal{H}$  can be obtained from the classical one (5.15) by the same procedure. In complete analogy to the quantum harmonic oscillator, which outputs  $H = \omega(a^\dagger a + 1/2)$ , we find (using (5.18) to write  $a(\mathbf{k})a^*(\mathbf{k}) \rightarrow a(\mathbf{k})a^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) + (2\pi)^3 2\omega_{\mathbf{k}}\delta^3(\mathbf{k} - \mathbf{k}')$ )

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \omega_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k}) + (\mathcal{E}_0 - \Omega_0)V \quad (5.20)$$

where  $V$  is the volume of space and

$$\mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3} \int d^3k \omega_{\mathbf{k}} \quad (5.21)$$

is the total zero-point energy of all the oscillators per unit volume  $V$  (where we interpreted  $V = (2\pi)^3 \delta^3(\mathbf{0})$ ). The innocuous  $1/2$  becomes a bit troubling when there are a continuous infinity of them. The constant  $\Omega_0$  was introduced precisely to get rid of this infinity; we simply choose  $\Omega_0 = \mathcal{E}_0$  to normalize our ground state energy to zero. (In quantum mechanics an overall shift to the Hamiltonian has no physical consequence, only energy differences are meaningful; gravity is another story.) Anyway, we see that the total Hamiltonian is indeed the “sum” (i.e. integral) of individual harmonic oscillator Hamiltonians, one for each momentum  $\mathbf{k}$ .

Recall that the classical Hamiltonian can also be written as  $H = \int d^3x T^{00}$ , as shown in (4.32). Similarly, we can write the total momentum as

$$P^j = \int d^3x T^{0j} = - \int d^3x \pi(\mathbf{x}) \partial^j \phi(\mathbf{x}) \quad \longrightarrow \quad \mathbf{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \mathbf{k} a(\mathbf{k})^\dagger a(\mathbf{k}). \quad (5.22)$$

The commutation relations (5.18) let us show<sup>12</sup>

$$[H, a^\dagger(\mathbf{k})] = \omega_{\mathbf{k}} a^\dagger(\mathbf{k}), \quad [H, a(\mathbf{k})] = -\omega_{\mathbf{k}} a(\mathbf{k}). \quad (5.23)$$

$$[\mathbf{P}, a^\dagger(\mathbf{k})] = \mathbf{k} a^\dagger(\mathbf{k}), \quad [\mathbf{P}, a(\mathbf{k})] = -\mathbf{k} a(\mathbf{k}). \quad (5.24)$$

With a quantum Hamiltonian in hand, the Hilbert space is simply the space of eigenvectors of this operator. But it is simpler to work with ladder operators. The vacuum is defined by

$$a(\mathbf{k})|0\rangle = 0 \quad \text{for all } \mathbf{k}, \quad \langle 0|0\rangle = 1. \quad (5.25)$$

In the same way that the creation operator  $a^\dagger$  builds up the Hilbert space of the quantum harmonic oscillator, here the family of  $a^\dagger(\mathbf{k})$  build up the Hilbert space in quantum field theory:

$$a^\dagger(\mathbf{k})|0\rangle = |k\rangle \implies H|k\rangle = \omega_{\mathbf{k}}|k\rangle, \quad \mathbf{P}|k\rangle = \mathbf{k}|k\rangle. \quad (5.26)$$

So  $|k\rangle$  is an eigenstate of the Hamiltonian  $H$  and momentum  $\mathbf{P}$ . It is interpreted as a state with a single particle with 4-momentum satisfying  $k^2 = -m^2$ , as before. We could have written  $|k\rangle$  as  $|\mathbf{k}\rangle$  to emphasize that the 3-vector is arbitrary but the 4-vector is not. Notice also that calling it a particle does not mean it is localized in space; after all, it is a momentum eigenstate!

We compute the normalization of 1-particle momentum eigenstates using (5.18):

$$\langle k'|k\rangle = \langle 0|a(\mathbf{k})a^\dagger(\mathbf{k}')|0\rangle = \langle 0|a^\dagger(\mathbf{k}')a(\mathbf{k})|0\rangle + (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}') = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}') \quad (5.27)$$

The resolution of the identity *in the sector of 1-particle states* is given by

$$\mathbf{1} = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} |k\rangle \langle k|. \quad (5.28)$$

This is checked by

$$\mathbf{1}|k'\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} |k\rangle \langle k|k'\rangle = \int d^3k |k\rangle \delta^3(\mathbf{k} - \mathbf{k}') = |k'\rangle. \quad (5.29)$$

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<sup>12</sup>This may be a bit confusing, since it looks like the operator  $a(\mathbf{k})$  (and  $a^\dagger(\mathbf{k})$ ) is time-independent, which would seem to suggest that  $[H, a(\mathbf{k})] = -i\partial_t a(\mathbf{k}) = 0$  through the Heisenberg equations of motion (3.12). But all operators are in the Heisenberg picture and have time-dependence, so  $a(\mathbf{k}) := a(\mathbf{k}, t = 0)$ .

We build  $n$ -particle states as

$$a^\dagger(\mathbf{k}_1) \cdots a^\dagger(\mathbf{k}_n)|0\rangle = |k_1, \dots, k_n\rangle. \quad (5.30)$$

These are eigenstates of  $H, \mathbf{P}$  with energy  $\omega_{\mathbf{k}_1} + \cdots + \omega_{\mathbf{k}_n}$  and momentum  $\mathbf{k}_1 + \cdots + \mathbf{k}_n$ . Notice a key difference from the interpretation in the case of the harmonic oscillator: there, applying multiple annihilation operators did not increase the number of particles. We only ever had one particle.

The overlaps between  $n$ -particle states and  $m$ -particle states vanishes for  $n \neq m$ . This means that the Hilbert space of quantum mechanics becomes a special sort of Hilbert space in quantum field theory called a Fock space:

$$\mathcal{F} = \oplus_n \mathcal{H}_n. \quad (5.31)$$

It is the direct sum of Hilbert spaces for  $n$  particles, over all  $n$ . For a single species of particle, states in each  $\mathcal{H}_n$  are linear combinations of states  $\{|k_1, \dots, k_n\rangle\}$  of all possible momenta  $k_i$  satisfying  $k_i^2 = -m^2$  and  $k_i^0 > 0$ . Abstractly, for bosonic theories as we considered above, each  $\mathcal{H}_n$  is a symmetrization of  $n$  copies of a single-particle Hilbert space, e.g. since  $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2) = a^\dagger(\mathbf{k}_2)a^\dagger(\mathbf{k}_1)$  we have  $|\psi_i, \psi_j\rangle_{n=2} = c_{ij}|\psi_i\rangle_{n=1} \otimes |\psi_j\rangle_{n=1} + c_{ji}|\psi_j\rangle_{n=1} \otimes |\psi_i\rangle_{n=1}$  for arbitrary  $c_{ij}$ . This structure is built into the Hilbert space we constructed above, and is true more generally. For fermionic Hilbert spaces each  $\mathcal{H}_n$  is an antisymmetrization of the single-particle Hilbert space.

Let's see what the field operator  $\phi(\mathbf{x}, t = 0)$  does by acting on the vacuum:

$$\phi(\mathbf{x})|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left( a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right) |0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}} |k\rangle. \quad (5.32)$$

This is a superposition of single-particle states with momentum  $\mathbf{k}$ . This is similar to what we see in nonrelativistic quantum mechanics when expressing  $|\mathbf{x}\rangle$  in the momentum basis. So we want to interpret  $\phi(x)$  as creating a particle at position  $\mathbf{x}$ . Indeed, taking an overlap with a single-particle momentum eigenstate, we find:

$$\langle p|\phi(\mathbf{x})|0\rangle = \langle 0|a(\mathbf{p}) \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left( a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right) |0\rangle \quad (5.33)$$

$$= \langle 0| \int d^3k e^{-i\mathbf{k}\cdot\mathbf{x}} \delta^3(\mathbf{k} - \mathbf{p}) |0\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}}. \quad (5.34)$$

This is the same (up to normalization) as the projection of a position eigenstate onto a

momentum eigenstate in nonrelativistic quantum mechanics:

$$\langle \mathbf{p} | \mathbf{x} \rangle = e^{-i\mathbf{p} \cdot \mathbf{x}} / \sqrt{2\pi}. \quad (5.35)$$

So we can identify  $\phi(\mathbf{x})|0\rangle = |\mathbf{x}\rangle$ , i.e.  $\phi(\mathbf{x})$  creates a particle at position  $\mathbf{x}$ . This is to be expected, since  $\phi$  has a structure similar to the operator which creates position eigenstates in quantum mechanics,  $x \sim a + a^\dagger$ . In the way that  $\phi(\mathbf{x})$  is analogous to  $x$ ,  $\pi(\mathbf{x})$  is analogous to  $p$ . But  $\pi(\mathbf{x})$  does not create states with a given physical momentum; instead, it also creates a state at position  $\mathbf{x}$ . Another interpretation of the above quantities is that the vacuum-to-one particle matrix elements of the field operator  $\phi$  give the plane-wave wavefunctions from nonrelativistic quantum mechanics for a particle in a momentum eigenstate. In this way, the quantum-mechanical limit of this quantum field theory is given by restricting to 1-particle states; see (2.82) - (2.87) of [4].

## 6 A hint of the spin-statistics theorem

If we wanted a theory of fermions in the previous section, then presumably we just replace the commutators by anticommutators. The problem with this, however, is that when we pass from the classical Hamiltonian to the quantum one, we get  $a^*(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^*(\mathbf{k}) \rightarrow a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) = (2\pi)^3 2\omega \delta^3(\mathbf{k} - \mathbf{k}')$ , which leads to  $H = (\mathcal{E}_0 - \Omega_0)V$ , a constant. This is a hint of the spin-statistics theorem, which we now prove for spin-zero particles.

The basic principle we will use is causality. In quantum mechanics observables that commute are simultaneously diagonalizable and hence simultaneously observable. They should therefore not be able to influence each other. In relativity, if points are spacelike separated then they should not be able to communicate, i.e. measurements at one point should not affect measurements at the other point. Together, this suggests the causality criterion, that observables at spacelike separation commute:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0, \quad (x - y)^2 > 0. \quad (6.1)$$

This actually can be shown to follow from Lorentz invariance of the S-matrix. The basic idea is then just that these observables are built out of the  $\phi$  fields in the Lagrangian which therefore have to commute. As we saw in (2.37), this commutativity leads to symmetric wavefunctions and therefore Bose-Einstein statistics. While this seems like it should force fermionic particles to commute as well, it turns out that observables are bilinear in spinors and have integer spin. This is the reason why causality cannot be used to prove the spin-statistics theorem for fermionic particles. (The theorem instead makes fermions anticommute, which

leads to antisymmetric wavefunctions and Fermi-Dirac statistics.)

To see this idea in practice, consider the theory of a charged scalar field considered before,  $\mathcal{L} = |\partial\phi|^2$ . The conserved current in (4.26) implies a charge density  $j^0 = i(\phi^\dagger\partial_0\phi - (\partial_0\phi^\dagger)\phi)$ . This is an observable, so we should have  $[j^0(\mathbf{x}, t = 0), j^0(\mathbf{y}, t = 0)] = 0$ . This is indeed true when the theory is quantized as in the previous section (where we now have two fields,  $\phi$  and  $\phi^\dagger$ ), since  $\phi(\mathbf{x}, t = 0)$  and  $\partial_0\phi(\mathbf{x}, t = 0)$  commute with  $\phi(\mathbf{y}, t = 0)$  and  $\partial_0\phi(\mathbf{y}, t = 0)$ . However, if we quantize the fields or the ladder operators with anticommutation relations, we will no longer find the charge density observable to commute at spacelike separation.

## 7 Correlation functions in quantum field theory

The fundamental measurable quantities in quantum mechanics and quantum field theory are correlation functions. Recall that in classical probability theory, for a random variable<sup>13</sup>  $Y(s)$  and a probability distribution  $P(s)$  over the space of possible outcomes  $s \in \mathbb{R}$ , we often compute the quantities

$$\langle Y^n \rangle := \int_{-\infty}^{\infty} ds P(s) Y(s)^n. \quad (7.1)$$

$Y(s)$  is a user-chosen function of the space of possible outcomes. The mean or expectation value  $\langle Y \rangle$  is given by  $n = 1$ , the variance is given by  $\langle Y^2 \rangle - \langle Y \rangle^2$ , and higher moments are given by higher  $n$ . When people refer to the mean or variance of the distribution  $P(s)$  instead of some function  $Y$ , they are implicitly referring to the case where  $Y(s) = s$ . In this case they will say that the random variable  $Y$  has probability distribution  $P(s)$ . This should fit your intuition, since  $Y = s$  and  $s$  has probability according to  $P(s)$ .

We can have our random variable depend on spacetime,  $Y(x)$ , in which case the computable quantities can be richer:

$$\langle Y(x_1) \cdots Y(x_n) \rangle = \int_{-\infty}^{\infty} ds P(s) Y(s, x_1) \cdots Y(s, x_n). \quad (7.2)$$

For  $n = 2$ , this measures how correlated  $Y(x_1)$  and  $Y(x_2)$  are (technically this is called the covariance and the correlation is a normalized version of this). Often we measure correlation functions (like in cosmology), and infer from them a probability distribution.

In quantum field theory the correlation functions can be defined very analogously once we introduce the path integral in the next section, but for now we will use our direct Hilbert

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<sup>13</sup>The term “random variable” is pretty confusing – think of it just as a function. To see why people call it “random” think of the case  $Y(s) = s$  and the simple set of outcomes of e.g. a die roll or a coin flip.

space picture. The expectation value of a quantum field in the vacuum is given by

$$\langle \phi(x) \rangle := \langle 0 | \phi(x) | 0 \rangle. \quad (7.3)$$

When we have angled brackets without a specified quantum state, we assume it to be the vacuum as in the above definition. This one-point function vanishes for the scalar field we constructed in the previous section. A two-point correlation function is given by

$$\langle \phi(x) \phi(y) \rangle := \langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (7.4)$$

We can compute this using (5.18):

$$\langle 0 | \int \frac{d^3 k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left( a(\mathbf{k}) e^{ikx} + a^\dagger(\mathbf{k}) e^{-ikx} \right) \int \frac{d^3 k'}{(2\pi)^3 2\omega_{\mathbf{k}'}} \left( a(\mathbf{k}') e^{ik'y} + a^\dagger(\mathbf{k}') e^{-ik'y} \right) | 0 \rangle \quad (7.5)$$

$$= \langle 0 | \int \frac{d^3 k}{(2\pi)^3 2\omega_{\mathbf{k}}} \frac{d^3 k'}{(2\pi)^3 2\omega_{\mathbf{k}'}} a(\mathbf{k}) e^{ikx} a^\dagger(\mathbf{k}') e^{-ik'y} | 0 \rangle \quad (7.6)$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega_{\mathbf{k}}} \frac{d^3 k'}{(2\pi)^3 2\omega_{\mathbf{k}'}} e^{ikx} e^{-ik'y} (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}') \langle 0 | 0 \rangle \quad (7.7)$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\sqrt{\mathbf{k}^2 + m^2}} e^{ik(x-y)}. \quad (7.8)$$

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*Example 1:* Let's evaluate the integral (7.8), considering the cases of purely timelike and purely spacelike separation separately. For purely timelike,  $\mathbf{x} = \mathbf{y}$  and  $x^0 - y^0 = t$ , and we define  $k := |\mathbf{k}|$  (notice the double-use of notation: this is the magnitude of the 3-momentum below, not the 4-momentum as it was before!) to get

$$\langle \phi(x) \phi(y) \rangle = \int \frac{k^2 \sin \theta dk d\theta d\phi}{(2\pi)^3 2\sqrt{k^2 + m^2}} e^{-it\sqrt{k^2 + m^2}} = \frac{1}{4\pi^2} \int_0^\infty \frac{dk k^2}{2\sqrt{k^2 + m^2}} e^{-it\sqrt{k^2 + m^2}} \quad (7.9)$$

$$(7.10)$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2} e^{-iEt} = -\frac{imK_1(imt)}{4\pi^2 t}, \quad (7.11)$$

where we changed variables from  $k = \sqrt{E^2 - m^2}$  to  $E$ . The integral was done with Mathematica, obtaining a modified Bessel function of the second kind  $K_1$ .<sup>14</sup> We can instead evaluate it by hand at large  $t$  by stationary phase. This method gives  $E \approx m - \frac{i}{2t}$  as the

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<sup>14</sup>To make the integral converge we need to make  $t$  slightly complex,  $t \rightarrow t - i\epsilon$ . This also means that the final answer should have  $t \rightarrow t - i\epsilon$ , which is needed to ensure the full correlator at arbitrary separation satisfies the equation of motion and not a Green's function equation. Without it there is a singularity as  $t^2 \rightarrow x^2 + y^2 + z^2$  which leads to a Green's function when acted upon with  $\square - m^2$ .

point of stationary phase, which implies (we include the prefactor in the stationary phase approximation)

$$\lim_{t \rightarrow \infty} \langle \phi(x)\phi(y) \rangle \sim \frac{\sqrt{m}}{t^{3/2}} e^{-imt}. \quad (7.12)$$

This scaling precisely agrees with the expansion of the Bessel function  $K_1$  at large  $t$ .

If we pick a purely spacelike separation,  $x^0 = y^0$  and  $\mathbf{x} - \mathbf{y} = \mathbf{r}$ , we find

$$\langle \phi(x)\phi(y) \rangle = \int \frac{k^2 \sin \theta dk d\theta d\phi}{(2\pi)^3} \frac{e^{ikr \cos \theta}}{2\sqrt{k^2 + m^2}} = \frac{1}{4\pi^2} \int_0^\infty dk \frac{k^2}{2\sqrt{k^2 + m^2}} \frac{e^{ikr} - e^{-ikr}}{ikr} \quad (7.13)$$

$$(7.14)$$

$$= \frac{-i}{8\pi^2 r} \int_{-\infty}^\infty dk \frac{k e^{ikr}}{\sqrt{k^2 + m^2}} \quad (7.15)$$

We evaluate this using complex analysis. This integrand has branch cuts beginning at  $k = \pm im$ ; we direct the branch cut at  $k = +im$  upward to  $i\infty$  and the one at  $k = -im$  downward to  $-i\infty$ . Since the integrand decays exponentially as  $k \rightarrow i\infty$ , we can close the contour to hug the branch cut in the upper half plane. We define  $\kappa = ik$  to write

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2 r} \int_m^\infty d\kappa \frac{\kappa e^{-\kappa r}}{\sqrt{\kappa^2 - m^2}} = \frac{m}{4\pi^2 r} K_1(mr). \quad (7.16)$$

We again could have evaluated the final integral by steepest descent, which would give  $\kappa \approx m - \frac{1}{2r}$  as the saddle point, leading to (we include the prefactor in the stationary phase approximation)

$$\lim_{r \rightarrow \infty} \langle \phi(x)\phi(y) \rangle \sim \frac{\sqrt{m}}{r^{3/2}} e^{-mr}. \quad (7.17)$$

We see that the field has nonvanishing correlation for (arbitrarily large) spacelike separation, but does go to zero, satisfying what is known as “cluster decomposition.”

One can also consider correlation functions in excited states  $\langle f|\phi(x)\phi(y)|f \rangle$ , or we can consider “overlaps”  $\langle f|\phi(x)\phi(y)|i \rangle$ , but often this is written in terms of a bigger correlator in the vacuum, e.g. in (5.33) we wrote the overlap  $\langle p|\phi(x)|0 \rangle$  as the vacuum correlator  $\langle 0|a(\mathbf{p})\phi(x)|0 \rangle$ . And of course we can (and will) consider higher-point functions  $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ .

The fact that  $\langle \phi(x)\phi(y) \rangle$  is nonzero for spacelike separation seems acausal. But the correct

measure of causality was discussed in the previous section, and is instead  $[\phi(x), \phi(y)] = 0$  for spacelike separation. The commutator is a pure number so we do not need to evaluate it in the vacuum; the fact that it is zero means it is zero in any state! For  $x_0 = y_0$  we can simply use the equal-time commutators to see that this vanishes, but it is true for arbitrary spacelike separation. To see this, let's evaluate the commutator in the vacuum,  $\langle [\phi(x), \phi(y)] \rangle = \langle \phi(x)\phi(y) \rangle - \langle \phi(y)\phi(x) \rangle$  and notice that  $\langle \phi(y)\phi(x) \rangle$  for spacelike separation can be computed exactly like  $\langle \phi(x)\phi(y) \rangle$  in the example above, since the method only depended on  $r = |\mathbf{x} - \mathbf{y}|$ . For timelike separation, however,  $\langle [\phi(x), \phi(y)] \rangle$  is nonzero. The fact that this object can be identically zero for a set of nonzero measure but not zero everywhere is because this object is a distribution, not an analytic function of  $x$  and  $y$ .

The two-point function  $\Delta(x - y) := \langle \phi(x)\phi(y) \rangle$  is sometimes called the Wightman function. It satisfies the Klein-Gordon equation:

$$(-\partial_x^2 + m^2)\Delta(x - y) = 0. \quad (7.18)$$

If we have a source on the right-hand-side, then we would solve the equation via the Green's function method (see Appendix C for a review). This means we need to first solve

$$(-\partial_x^2 + m^2)G(x - y) = \delta^4(x - y). \quad (7.19)$$

Fourier transforming

$$G(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \tilde{G}(k) \quad (7.20)$$

gives

$$(k^2 + m^2)\tilde{G}(k) = 1 \implies G(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2 + m^2}. \quad (7.21)$$

This is the massive generalization of (C.40). We need to integrate this over  $k^0 \in \mathbb{R}$ . But there are singularities at  $k^2 + m^2 = -(k^0)^2 + \mathbf{k}^2 + m^2 = 0$ , i.e. there are simple poles at  $k^0 = \pm\omega_{\mathbf{k}} = \pm\sqrt{\mathbf{k}^2 + m^2}$ . To define the integral, we need to dodge these poles by a small excursion upward or downward into the complex plane. The two choices per pole gives four possible definitions for this integral. Going above both poles defines the **retarded propagator**:

$$G_R(x - y) := i\theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle. \quad (7.22)$$

To recover this expression and analogous ones below, close the  $k^0$  contour downward for  $x^0 > y^0$  and upward otherwise; the two terms of the commutator come from the two poles. Recall from Appendix C that the retarded propagator results from fixing the field and its time derivative at some time (in this case at  $x^0 = y^0$ ) and tells you how to propagate into the future ( $x^0 > y^0$ ).

Going below both poles defines the **advanced propagator**:

$$G_A(x - y) := i\theta(y^0 - x^0)\langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (7.23)$$

The advanced propagator results from fixing the field and its time derivative at some time (in this case at  $x^0 = y^0$ ) and tells you how to propagate into the past ( $x^0 < y^0$ ).

Going below the pole  $k^0 = -\omega_{\mathbf{k}}$  and above the pole  $k^0 = +\omega_{\mathbf{k}}$  defines the time-ordered or **Feynman propagator** (doing the opposite defines the anti-Feynman propagator  $G_{AF}$ ):

$$G_F(x - y) = \begin{cases} i\Delta(x - y) & \text{for } x^0 > y^0 \\ i\Delta(y - x) & \text{for } x^0 < y^0 \end{cases} \quad (7.24)$$

We can write this by defining the time-ordering symbol  $T$ , which means we should put the operators in order from left to right beginning with the latest time on the left (the anti-Feynman propagator has an anti-time-ordering symbol, which puts operators in the opposite order):

$$G_F(x - y) = i\langle 0 | T\phi(x)\phi(y) | 0 \rangle := i\theta(x^0 - y^0)\langle 0 | \phi(x)\phi(y) | 0 \rangle + i\theta(y^0 - x^0)\langle 0 | \phi(y)\phi(x) | 0 \rangle \quad (7.25)$$

This prescription is also often written as

$$G_F(x - y) := \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon}, \quad (7.26)$$

because then the poles are at  $k^0 = -\omega_{\mathbf{k}} + i\epsilon/(2\omega_{\mathbf{k}})$  and  $k^0 = +\omega_{\mathbf{k}} - i\epsilon/(2\omega_{\mathbf{k}})$ . We can therefore run the  $k^0$  contour exactly along the real line, which naturally goes below the first pole and above the second one. When closing the contour, we only ever pick up one of the two poles, which is why we only ever get the correlator  $\Delta$  and not the commutator  $[\phi(x), \phi(y)]$ .

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*Example 2:* Let's check that  $G_F(x - y)$  solves the Green's function equation:

$$(-\partial_x^2 + m^2)G_F(x - y) = \delta^4(x - y). \quad (7.27)$$

We will use the representation (7.25) and the fact that the two-point function  $\Delta$  satisfies the Klein-Gordon equation (7.18). The first term in  $G_F$  gives (we drop the  $x$  subscript on the derivative to reduce clutter)

$$i [-\partial^2 \theta(x^0 - y^0)] \Delta(x - y) - 2i \partial^\mu \Theta(x^0 - y^0) \partial_\mu \Delta(x - y) - i \theta(x^0 - y^0) (\partial^2 - m^2) \Delta(x - y) \quad (7.28)$$

$$= -i \delta'(x^0 - y^0) \langle \phi(x) \phi(y) \rangle - 2i \delta(x^0 - y^0) \langle \pi(x) \phi(y) \rangle = -i \delta(x^0 - y^0) \langle \pi(x) \phi(y) \rangle \quad (7.29)$$

where we integrated by parts in the last step (since these expressions are formally defined under integrals). The second term in  $G_F$  gives

$$i [-\partial^2 \theta(y^0 - x^0)] \Delta(y - x) - 2i \partial^\mu \theta(y^0 - x^0) \partial_\mu \Delta(y - x) = i \delta(x^0 - y^0) \langle \phi(y) \pi(x) \rangle \quad (7.30)$$

The full expression is therefore

$$(-\partial^2 + m^2) G_F = -i \delta(x^0 - y^0) \langle [\pi(x), \phi(y)] \rangle = \delta^4(x - y). \quad (7.31)$$

Notice in the last line we could evaluate the commutator at  $x^0 = y^0$  due to the delta function. The calculations for  $G_{AF}$ ,  $G_R$  and  $G_A$  are similar.

For more on the retarded and advanced propagators, and how they result from choices of boundary conditions in the physical problem of interest, see the beginning of Appendix C.

It was pretty straightforward to compute the correlation function and various Green's functions built out of them in our free field theory. But we would like to consider more complicated theories, which can have nonlinear self-interactions and many species of particles. An example of using the Green's function method to compute the solution to a nonlinear equation of motion, perturbatively in the nonlinearity, is given in Appendix C. This procedure was exhibited in a purely classical field theory and was shown to be encapsulated in a set of Feynman diagrams. To compute correlation functions in quantum field theory we will set up a similar machinery, generalized to include quantum fluctuations. The primary new ingredient is that the lines in the classical Feynman diagrams can now close in on themselves.

While we can set up a Feynman diagrammatic expansion to calculate any of  $\Delta$ ,  $G_R$ ,  $G_A$ ,  $G_F$ , or  $G_{AF}$ , we will see in Section 11 that  $G_F$  is the relevant object to eventually compute scattering amplitudes. So we will orient our diagrammatic expansion around computing  $G_F$ . The quickest way to derive the Feynman diagrams is through the path integral representation of quantum mechanics, which we now turn to.

## 7.1 Where did the wavefunction go?

In nonrelativistic quantum mechanics, we are obsessed with the wavefunction. It is given by taking the overlap  $\langle x|\psi(t)\rangle =: \psi(x,t)$  between a state in the Hilbert space  $|\psi(t)\rangle$  and a position eigenstate  $|x\rangle$ . It evolves via the Schrödinger equation,  $i\hbar\partial_t\psi(x,t) = H\psi(x,t)$ . This gives us a physical picture through the probability  $|\psi(x,t)|^2$  for the particle to be found at position  $x$  at time  $t$ . For a time-independent Hamiltonian, we are instructed to compute the eigenvectors and eigenvalues of the Hamiltonian, from which we can construct a general state in the Hilbert space and calculate its time evolution. Notice this is all in the Schrödinger picture, since the state  $|\psi(t)\rangle$  depends on time. This is a useful framework since it is closely connected to easily measurable quantities. Take the hydrogen atom, with its various orbitals and energy levels. These are easily observable, say through photon emission. And the probability amplitude  $|\psi(x,t)|^2$  gives us a physical picture of where the electron is likely to be.

Since quantum field theory is just (relativistic) quantum mechanics, the wavefunction and its interpretation continue to exist, only slightly modified. The relevant quantum variables are now the fields and their conjugate momenta, so we upgrade the wavefunction  $\psi(x,t) := \langle x|\psi(t)\rangle$  to the wavefunctional  $\Psi[\phi;t] := \langle\phi|\Psi(t)\rangle$ . The object  $|\phi\rangle$  is a field eigenstate,  $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$ , exactly analogous to the position eigenstate  $|x\rangle$ . There is such an eigenstate for every given field configuration  $\phi$ . Instead of computing the probability to find the particle at some location  $x$ , the natural quantity is the probability that the field has the configuration  $\phi$ . This wavefunctional evolves according to the functional Schrödinger equation

$$i\hbar\partial_t\Psi[\phi;t] = H\Psi[\phi;t] \tag{7.32}$$

where the representation of the momentum density is given by  $\pi(x) = -i\hbar\frac{\delta}{\delta\phi(x)}$ . While this structure exists in quantum field theory, it is not discussed in many textbooks, which tend to focus more on computing quantities relevant for particle phenomenology, i.e. S-matrix elements. For these, the Heisenberg picture where the states do not evolve in time tends to be more natural, due to the central role played by expectation values of time-ordered operators. You will see that texts written by condensed matter physicists show the Schrödinger picture more love.

So, whether quantum mechanics or quantum field theory, the structure of the theory allows the calculation of eigenvectors/eigenvalues of the Hamiltonian, correlation functions of time-dependent operators, or time-dependent wavefunctions/wavefunctionals. Which we choose to focus on is primarily a function of what is easily measurable in experiments. The

physical picture of a universe described by quantum field theory, however, remains that of a state in a Hilbert space, which can give probability amplitudes for observing certain field configurations or their conjugate momenta (but not both!).

## 8 Path integrals in quantum mechanics

The path integral is a formulation of quantum mechanics (and quantum field theory) which is precisely equivalent to the canonical formalism we have developed so far. A heuristic understanding of the path integral is given by considering the double-slit experiment. The explanation for the interference pattern is that the particle goes through both slits as long as you're not looking. But now you can imagine putting another screen with two more slits. The particle should go through  $2 \times 2 = 4$  possible paths. More screens with more slits leads to even more paths. Now imagine punching more slits per screen – that leads to many more paths! We can imagine punching so many slits that the screens disappear altogether, and we are led to the conclusion that the particle takes every possible path. This is the intuition behind the path integral, but making it precise will take some work.

Let's consider a generic quantum-mechanical Hamiltonian for a nonrelativistic particle:

$$H(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}). \quad (8.1)$$

We want to compute the amplitude for the particle at position  $q_0$  at time  $t_0$  to get to position  $q_f$  at time  $t_f$ . This is given by

$$\langle q_f | e^{-iH(t_f - t_0)} | q_0 \rangle. \quad (8.2)$$

We can split time up into  $N + 1$  equal segments of length  $\delta t = \frac{t_f - t_0}{N + 1}$ . After evolving for one of these segments, we insert a complete set of states  $1 = \int dq |q\rangle \langle q|$ . We do this  $N$  times since we have  $N + 1$  segments and we get:

$$\int \prod_{k=1}^N dq_k \langle q_f | e^{-iH\delta t} | q_N \rangle \langle q_N | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | \cdots | q_1 \rangle \langle q_1 | e^{-iH\delta t} | q_0 \rangle. \quad (8.3)$$

All we did is stick in the identity a bunch of times. The integrals range over all possible real values. To split up the propagator  $e^{iH\delta t}$  into momentum and position pieces we need to use the Baker-Campbell-Hausdorff formula (technically the Zassenhaus formula) for operators

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{\frac{1}{6}(2[B,[A,B]] + [A,[A,B]])} \dots \quad (8.4)$$

Working to linear order in  $\delta t$ , however, lets us ignore all the noncommutative junk to write<sup>15</sup>

$$\langle q_1 | e^{-iH\delta t} | q_0 \rangle \approx \langle q_1 | e^{-i\frac{p_0^2}{2m}\delta t} e^{-iV(\hat{q})\delta t} | q_0 \rangle = \langle q_1 | \int dp_0 e^{-i\frac{p_0^2}{2m}\delta t} | p_0 \rangle \langle p_0 | e^{-iV(\hat{q})\delta t} | q_0 \rangle \quad (8.5)$$

$$= \int \frac{dp_0}{2\pi} e^{-i\frac{p_0^2}{2m}\delta t} e^{-iV(q_0)\delta t} e^{ip_0(q_1 - q_0)} = \int \frac{dp_0}{2\pi} e^{-iH(p_0, q_0)\delta t} e^{ip_0(q_1 - q_0)}, \quad (8.6)$$

where we inserted a complete set of momentum eigenstates and used  $\langle q | p \rangle = e^{ipq}/\sqrt{2\pi}$ . Our amplitude (8.3) therefore becomes

$$\int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} e^{-iH(p_j, q_j)\delta t}. \quad (8.7)$$

Defining  $q_{j+1} - q_j = \dot{q}_j \delta t$  and taking  $\delta t \rightarrow 0$ ,  $N \rightarrow \infty$  lets us write

$$\int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{i(p_j \dot{q}_j - H(p_j, q_j))\delta t} \longrightarrow \int Dq Dp \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_f} dt (p\dot{q} - H(p, q)) \right). \quad (8.8)$$

We have reintroduced  $\hbar$  in the final step. The limit of the product of integrals has become a **path integral**, with measure  $Dq Dp$ . This path integral instructs us to integrate over all paths in phase space with  $q(t_0) = q_0$  (and arbitrary initial momentum) and  $q(t_f) = q_f$  (and arbitrary final momentum). It is annoying but conventional to not explicitly state the boundary conditions of the path integral in the formula.

This integrand looks tantalizingly close to  $e^{iS}$  for action  $S = \int dt L(q, \dot{q})$ . To make this connection precise we need to get rid of the momenta somehow. In the cases we consider the Hamiltonian will be quadratic in momentum, for which we can do the path integral over the momenta exactly. See Appendix D for a review of Gaussian integrals evaluated exactly and by saddle point. Here we will evaluate the path integral over the momenta by saddle point:

$$\partial_p(p\dot{q} - H(p, q))|_{p_\star} = 0 \implies \dot{q} = \frac{\partial H(p_\star, q)}{\partial p_\star}. \quad (8.9)$$

This is just Hamilton's equation! This means that solving for  $p_\star$  in terms of  $q, \dot{q}$  and using this to approximate the path integral over  $p$  in (8.8) implements the Legendre transformation

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<sup>15</sup>For Hamiltonians that have products of positions and momenta, a choice needs to be made about their ordering when passing to the quantum theory. For the choice of "Weyl ordering" which keeps them symmetric see Chapter 6 of [1].

from Hamiltonian to Lagrangian (cf (3.5)) to give

$$\boxed{\langle q_f | e^{-iH(t_f-t_0)} | q_i \rangle = \int Dq \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_f} dt L(q(t), \dot{q}(t)) \right) = \int Dq e^{\frac{i}{\hbar} S[q]}. \quad (8.10)}$$

This saddle point approximation is good for small  $\hbar$ , whereas the phase space path integral we began with is true for general  $\hbar$ . In the case where the integral over the momentum is Gaussian the path integral over momenta is exactly calculable (i.e. for any  $\hbar$ ) and agrees with the saddle point approximation up to some computable prefactors that can be absorbed into the definition of the measure  $Dq$ . Actually this is true even for more general Hamiltonians, see Polchinski String Theory Vol. 1 Appendix A. So this Lagrangian path integral is quite general.

**The factor of  $\hbar$  shows us how the classical equations are recovered. For  $\hbar \rightarrow 0$ , we can evaluate this path integral by stationary phase, obtaining**

$$\frac{\delta S}{\delta q} = 0 \implies \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}. \quad (8.11)$$

**These are just the Euler-Lagrange equations. Taking the path integral as fundamental, this provides an explanation for why classical solutions obey the principle of least action!**

The path integral is powerful enough to calculate correlation functions as well. It is a little simpler for this to write the above overlap in the Heisenberg picture, where the operator  $\hat{q}(t) = e^{iHt} \hat{q} e^{-iHt}$  and we define an instantaneous eigenstate  $\hat{q}(t)|q, t\rangle = q|q, t\rangle$ . This eigenstate can be written as  $|q, t\rangle = e^{iHt}|q\rangle$ . (Don't confuse this with the time evolution of a Schrödinger picture quantum state, which comes with the opposite sign; we are in the Heisenberg picture, so the instantaneous eigenstates are time-independent states!) The above overlap we calculated was, in the Heisenberg picture,  $\langle q_f, t_f | q_i, t_i \rangle$ . But now let's compute the one-point function

$$\langle q_f, t_f | \hat{q}(t) | q_0, t_0 \rangle = \langle q_f | e^{iHt_f} (e^{iHt} \hat{q} e^{-iHt}) e^{iHt_0} | q_0 \rangle \quad (8.12)$$

$$= \langle q_f | e^{-iH(t_f-t)} \hat{q} e^{-iH(t-t_0)} | q_0 \rangle \quad (8.13)$$

$$= \int dq \langle q_f | e^{-iH(t_f-t)} | q \rangle \langle q | e^{-iH(t-t_0)} | q_0 \rangle q \quad (8.14)$$

$$= \int dq \left( \int_{q,t}^{q_f, t_f} Dp Dq e^{iS} \right) \left( \int_{q_0, t_0}^{q, t} Dp Dq e^{iS} \right) q \quad (8.15)$$

where  $t_0 < t < t_f$  and  $S = \int dt (p\dot{q} - H)$ . So we propagate up to time  $t$ , insert  $q$ , and continue

propagating to  $t_f$ . In the same way that  $p_j$  and  $q_j$  became  $p(t)$  and  $q(t)$  in (8.16), we will have  $q$  become  $q(t)$  in the path integral. This is because the above expression says we do the path integral of  $e^{iS}$  over all times  $t'$  between  $t_0$  and  $t_f$ , except when  $t' = t$ , in which case the integral is over  $e^{iS}q$ . If we could split off the integral over time  $t_0 < t' < t_f$  then this would be like a delta function insertion  $\delta(t' - t)q(t')$ . Anyway, we can write the final expression as<sup>16</sup>

$$\langle q_f, t_f | \hat{q}(t) | q_0, t_0 \rangle = \int Dq q(t) e^{iS} \quad (8.16)$$

where  $S = \int_{t_0}^{t_f} dt L$ . To understand the RHS of (8.16) it may help to go from there to (8.15) by discretizing the path integral as in (8.8).

Things get interesting when we compute a two-point function. For  $t_f > t_2 > t_1 > t_0$ , repeating the above steps gives

$$t_2 > t_1 : \quad \int Dq q(t_2)q(t_1) e^{iS} = \langle q_f, t_f | \hat{q}(t_2)\hat{q}(t_1) | q_0, t_0 \rangle. \quad (8.17)$$

However, notice the LHS is symmetric under  $t_1 \leftrightarrow t_2$  whereas the RHS is not. So this cannot be the general answer for arbitrary  $t_1, t_2$ . To see why, let's consider  $t_2 < t_1$  and insert complete sets of states to get

$$\langle q_f, t_f | \hat{q}(t_2)\hat{q}(t_1) | q_0, t_0 \rangle = \langle q_f | e^{-iH(t_f-t_2)} \hat{q} e^{-iH(t_2-t_1)} \hat{q} e^{-iH(t_1-t_0)} | q_0 \rangle \quad (8.18)$$

$$= \int dq_2 dq_1 \langle q_f | e^{-iH(t_f-t_2)} | q_2 \rangle \langle q_2 | e^{-iH(t_2-t_1)} | q_1 \rangle \langle q_1 | e^{-iH(t_1-t_0)} | q_0 \rangle q_2 q_1 \quad (8.19)$$

$$= \int dq_1 dq_2 \int_{q_2, t_2}^{q_f, t_f} Dq e^{iS} \int_{q_1, t_1}^{q_2, t_2} Dq e^{iS} \int_{q_0, t_0}^{q_1, t_1} Dq e^{iS} q_1 q_2 \quad (8.20)$$

Things look a bit weird. We have to propagate from  $t_0$  *past*  $t_2$  to  $t_1$ , insert  $q_1$ , propagate backward in time to  $t_2$ , insert  $q_2$ , and then propagate back up and past  $t_1$  to  $t_f$ . Actually, this is a totally legitimate expression (sometimes called timefolds), but we cannot eliminate the  $dq_1$  and  $dq_2$  integrals as before to collapse this into a single path integral!<sup>17</sup> For  $t_2 > t_1$ , this expression can be collapsed into a single path integral as above.

We thus see that the natural object computed by the path integral is the time-ordered

<sup>16</sup>Since we will only be interested in correlation functions of the field operators (and not their conjugate momenta), we can work with the Lagrangian path integral over  $Dq$  instead of the phase space one over  $Dp Dq$ , since the path integral over momenta will be unaffected by any insertions and can be evaluated by saddle point as before.

<sup>17</sup>We could write this as a single path integral that evolves backward in time, from  $t_f$  to  $t_2$  to  $t_1$  to  $t_0$ . This is totally fine, but if we compute  $n$ -point functions, then we can only capture two time-orderings by using this trick (i.e. we can compute time-ordered or anti-time-ordered correlators), whereas there are  $n!$  possible time-orderings.

correlation function or Feynman propagator:

$$\int Dq q(t_2)q(t_1) e^{iS} = \langle q_f, t_f | T[\hat{q}(t_2)\hat{q}(t_1)] | q_0, t_0 \rangle. \quad (8.21)$$

The earlier field will always come out to the right of the later field. The time-ordered correlator is also symmetric under  $t_1 \leftrightarrow t_2$ , as required. As already hinted, the input into the LSZ reduction formula for scattering amplitudes is the time-ordered correlator, so this formula is a coup. The generalization to many insertions is

$$\boxed{\int Dq q(t_1) \cdots q(t_n) e^{iS} = \langle q_f, t_f | T[\hat{q}(t_1) \cdots \hat{q}(t_n)] | q_0, t_0 \rangle.} \quad (8.22)$$

Another trick to calculate time-ordered correlators is the following. We imagine turning on an “external” or “classical” source  $f(t)$  (meaning it is a variable we do not path integrate over) which appears in the Lagrangian as

$$L(q, \dot{q}) \rightarrow L(q, \dot{q}) + f(t)q(t). \quad (8.23)$$

We then have the formulas

$$\frac{1}{i} \frac{\delta}{\delta f(t_1)} \langle q_f, t_f | q_0, t_0 \rangle_f = \int Dq q(t_1) e^{i \int dt (L+fq)} \quad (8.24)$$

$$\frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \langle q_f, t_f | q_0, t_0 \rangle_f = \int Dq q(t_1) \cdots q(t_n) e^{i \int dt (L+fq)} \quad (8.25)$$

Thus, to compute the time-ordered correlator of an arbitrary number of operators, we act with these functional derivatives and in the end set  $f = 0$ :

$$\boxed{\begin{aligned} \langle q_f, t_f | T[\hat{q}(t_1) \cdots \hat{q}(t_n)] | q_0, t_0 \rangle &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \langle q_f, t_f | q_0, t_0 \rangle \Big|_{f=0} \\ &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \int Dq e^{i \int_{t_0}^{t_f} dt (L+fq)} \Big|_{f=0}. \end{aligned}} \quad (8.26)$$

Notice also that in our units where  $\hbar = c = 1$ ,  $f$  has dimensions of  $1/\text{length}^2$ , which is the same as force in these units. So  $f$  has the interpretation of a force, which will become clearer in the harmonic oscillator example in Section 8.1.

## Path integrals and probabilities

Let's say that instead of  $e^{i(\dots)}$  in the above expressions we had real quantities like  $e^{-(\dots)}$  (this can be made precise by implementing a "Wick rotation"  $t \rightarrow it$ ). Then we can think of our path integral  $\int Dq e^{-S}$  as a probability distribution over possible paths  $q(t)$ . This identifies our correlators (8.22) as the path integral version of correlators from probability theory (7.2). The formulas are conceptually identical. From this perspective we can also understand  $\int Dq e^{-\int dt(L+fq)} = \int Dq e^{-S} e^{-\int dt fq}$  in (8.26) as the **moment generating function** (known as the probability generating function in the discrete case) of the probability distribution. For a continuous random variable  $X$  taking values over the reals with probability density  $P(x)$ , this is usually defined as

$$Z(s) = \int_{-\infty}^{\infty} dx P(x) e^{sx}. \quad (8.27)$$

The moments are computed from this formula by suitable derivatives

$$\langle x^n \rangle = \left( \frac{d}{ds} \right)^n Z(s) \Big|_{s=0} = \left( \frac{d}{ds} \right)^n \int_{-\infty}^{\infty} dx P(x) e^{sx} \Big|_{s=0}. \quad (8.28)$$

This is precisely analogous to (8.26).

Given this correspondence with probability distributions, which required an unexplained Wick rotation to make sure we had real probabilities, can we simulate quantum mechanics using Monte Carlo methods (i.e. sampling the distribution)? Yes! In quantum field theory that often occurs in the subject of lattice gauge theory, which basically samples the probability distribution  $e^{-S}$  (we need to discretize the space of field configurations to stand a chance, hence the lattice).

## Ground-state correlation functions

We have calculated time-ordered correlation functions between nontrivial initial and final quantum states,  $|q_0\rangle$  and  $|q_f\rangle$ . These choices entered into the boundary conditions of the path integral through the boundary conditions  $q(t_0) = q_0$ ,  $q(t_f) = q_f$ . What if we wanted initial and final states  $|i\rangle$ ,  $|j\rangle$  that are not position eigenstates? As long as we know their position-space wavefunctions we can write e.g.

$$|q_0\rangle \longrightarrow |i\rangle = \int dq_0 |q_0\rangle \langle q_0|i\rangle = \int dq_0 \psi_i(q_0) |q_0\rangle \quad (8.29)$$

In other words we can multiply our position eigenstate by the appropriate wavefunction and integrate, in addition to the rest of the path integral procedure outlined in this section. These integrals are hard to carry out to accommodate generic states, but if we are interested in the vacuum state  $|0\rangle$  then we can use a trick. The basic idea is very simple. We write our instantaneous position eigenstate in a basis of energy eigenstates

$$|q_0, t_0\rangle = e^{iHt_0}|q\rangle = e^{iHt_0} \sum_{n=0}^{\infty} |n\rangle \langle n|q_0\rangle = \sum_{n=0}^{\infty} e^{iE_n t_0} \psi_n^*(q_0) |n\rangle. \quad (8.30)$$

Let's assume our Hamiltonian is normalized so that the energies are nonnegative and  $E_0 = 0$ . Now if we take  $t_0 \rightarrow i\infty$ , we see that we have a sum of exponential suppressions which become very large, except for  $n = 0$ . All states are projected out except the vacuum. This is usually implemented by instead continuing  $t_0 \rightarrow (1 - i\epsilon)t_0$  then taking  $t_0 \rightarrow -\infty$ . Similarly, if we have  $\langle q_f, t_f| = \langle q_f|e^{-iHt_f}$  we can continue  $t_f \rightarrow (1 - i\epsilon)t_f$  and take  $t_f \rightarrow +\infty$  to pick out the vacuum.

This is sometimes summarized, in the Schrödinger picture, by  $\lim_{t \rightarrow (1-i\epsilon)\infty} e^{-iHt}|i\rangle \propto |0\rangle$ , i.e. infinitely long imaginary time evolution on an arbitrary state picks out the vacuum.

This trick means that we can calculate vacuum correlators with basically any initial and final conditions for our path integral, as long as (a) we evolve with a slightly complexified time  $t \rightarrow (1 - i\epsilon)t$  and (b) we evolve for infinitely long, i.e. we take  $t_0 \rightarrow -\infty$  and  $t_f \rightarrow +\infty$ . Performing this in our phase space path integral gives

$$\langle 0|0\rangle_f = \lim_{T \rightarrow (1-i\epsilon)\infty} \int Dp Dq \exp \left[ i \int_{-T}^T dt (p\dot{q} - H + fq) \right]. \quad (8.31)$$

We could equivalently write this as [1] does (cf (6.21)) by pulling the complexification from the limits of integration into the measure and integrand through  $t \rightarrow (1 - i\epsilon)t$ :

$$\langle 0|0\rangle_f = \int Dp Dq \exp \left[ i \int_{-\infty}^{\infty} dt (p\dot{q} - (1 - i\epsilon)H + fq) \right]. \quad (8.32)$$

Notice the  $(1 - i\epsilon)$  that appears from the measure  $dt$  cancels against a  $(1 - i\epsilon)^{-1}$  in  $p\dot{q}$ . We also redefined our source  $f$  to absorb this factor (which does not affect anything since it was arbitrary to begin with). So the only term left with a  $(1 - i\epsilon)$  is the Hamiltonian. We don't want to write a bunch of expressions with  $(1 - i\epsilon)$  in the integrands like (8.32), and we don't want to have to explicitly write the contour (8.31) every time, so we will use the following

notation:

$$\langle 0|0\rangle_f = \int Dp Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} (p\dot{q} - H + fq) \right]. \quad (8.33)$$

The measure  $\widetilde{dt}$  means we should use the slightly complexified contour of (8.31). We can do the momentum path integral and rewrite this in terms of a Lagrangian path integral

$$\langle 0|0\rangle_f = \int Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} (L(q, \dot{q}) + fq) \right]. \quad (8.34)$$

Time-ordered correlation functions in this ground state are obtained by applying (8.26):

**Ground state correlators**

$$\begin{aligned} \langle 0|T[\hat{q}(t_1) \cdots \hat{q}(t_n)]|0\rangle &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \int Dq e^{i \int_{-\infty}^{\infty} \widetilde{dt} (L+fq)} \Big|_{f=0} \\ &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \langle 0|0\rangle_f \Big|_{f=0}. \end{aligned} \quad (8.35)$$

## Interactions

It helps to split the Hamiltonian as  $H = H_0 + \lambda H_{\text{int}}$ .  $H_0$  is the “free” Hamiltonian, quadratic in the momenta and positions so that the path integrals can be done exactly.  $H_{\text{int}}$  is sometimes called the “interaction” Hamiltonian, and we cannot do the path integral over  $H_{\text{int}}$  exactly. (For simplicity we will assume it only depends on the position  $q$ , although the generalization is straightforward.) The parameter  $\lambda$  is a coupling constant, and we will assume that it is small and perform calculations perturbatively in  $\lambda$ . In the phase space path integral, the way this works is as follows. We write

$$\langle 0|0\rangle_f = \int Dp Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} (p\dot{q} - H_0(p, q) - \lambda H_1(q) + fq) \right] \quad (8.36)$$

$$= \exp \left[ -i\lambda \int_{-\infty}^{\infty} \widetilde{dt} H_1 \left( \frac{1}{i} \frac{\delta}{\delta f} \right) \right] \int Dp Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} (p\dot{q} - H_0(p, q) + fq) \right]. \quad (8.37)$$

The exponential on the outside is a functional differential operator, and by acting with it on the path integral part we will convert the argument of  $H_1$  into  $q$ . This formula becomes practical when we (a) perform the path integral exactly, which we can do since  $H_0$  is the free Hamiltonian and therefore it is a Gaussian path integral, (b) expand the exponential outside in a series in  $\lambda$ , which should give a good approximation with a few terms if  $\lambda$  is small. For  $H_0 = ap^2 + bq + cq^2$  and if we are only concerned with computing correlators of the position

operator, we can write the Lagrangian path integral

$$\langle 0|0\rangle_f = \exp \left[ i\lambda \int_{-\infty}^{\infty} \widetilde{dt} L_1 \left( \frac{1}{i} \frac{\delta}{\delta f(t)} \right) \right] \int Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} (L_0(q, \dot{q}) + fq) \right], \quad (8.38)$$

where  $L_1(q) = -H_1(q)$ .

## 8.1 Path integral for the harmonic oscillator

Let's do the simplest quantum-mechanical example, the harmonic oscillator. We want to compute ground-state overlap  $\langle 0|0\rangle_f$  so we will use the slightly complexified contour above to project to the vacuum. When we calculate the two-point function, it will be time-ordered since we are using the path integral,  $\langle 0|T(q(t_1)q(t_2))|0\rangle$ . But this is just the Feynman propagator, whose contour integral representation (7.26) has explicit  $i\epsilon$  factors. Therefore it should come as no surprise that the  $i\epsilon$  in the slightly complexified time  $(1 - i\epsilon)t$  feeds in exactly to become this  $i\epsilon$  factor in the Feynman propagator! Tracking the factors of  $\epsilon$  is a bit ugly so feel free to mentally set  $\epsilon = 0$  in a first pass.

We have the harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \quad (8.39)$$

We will set  $m = 1$  and write the Lagrangian generating functional

$$\langle 0|0\rangle_f = \int Dq \exp \left[ i \int_{-\infty}^{\infty} \widetilde{dt} \left( \frac{\dot{q}^2}{2} - \frac{\omega^2}{2}q^2 + fq \right) \right]. \quad (8.40)$$

Notice that the Euler-Lagrange equations following from this Lagrangian identify  $f$  as a driving force, as alluded to in the previous section.

Pulling the complexification of the contour into the measure and integrand gives

$$\langle 0|0\rangle_f = \int Dq \exp \left[ i \int_{-\infty}^{\infty} dt \left( (1 + i\epsilon) \frac{\dot{q}^2}{2} - (1 - i\epsilon) \frac{\omega^2}{2}q^2 + fq \right) \right], \quad (8.41)$$

where we redefine the (arbitrary) source  $f$  to absorb  $(1 - i\epsilon)$ . While we should be confident in Gaussian (path) integrals, the  $\dot{q}$  is a bit annoying, so let's Fourier transform. Plugging

$$g(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{g}(E) \quad (8.42)$$

into (8.41) for the functions  $q(t)$  and  $f(t)$  lets us write the term in square brackets in (8.41)

as

$$\frac{i}{2} \int dt \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{dE'}{2\pi} e^{-i(E+E')t} \left[ -(1+i\epsilon)EE' - (1-i\epsilon)\omega^2 \right] \tilde{q}(E)\tilde{q}(E') + 2\tilde{f}(E)\tilde{q}(E') \quad (8.43)$$

We can write  $2\tilde{f}(E)\tilde{q}(E') = \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)$  since the integrals of  $(e^{-i(E+E')t} \times)$  these two terms is identical due to an  $E \leftrightarrow E'$  symmetry (this will help below). Performing the  $t$  integral gives  $2\pi\delta(E+E')$ , and then performing the  $E'$  integral sets  $E' = -E$ . Altogether this gives

$$\frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left[ ((1+i\epsilon)E^2 - (1-i\epsilon)\omega^2) \tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E) \right]. \quad (8.44)$$

The coefficient of the quadratic term in  $\tilde{q}$  is  $E^2 - \omega^2 + i(E^2 + \omega^2)\epsilon$ ; we can redefine  $\epsilon$  to write this as  $E^2 - \omega^2 + i\epsilon$ . (8.44) is up in the exponential, so to do the path integral we need to complete the square to get rid of the terms linear in  $\tilde{q}$ . This can be done by redefining

$$\tilde{q}(E) = \tilde{x}(E) - \frac{\tilde{f}(E)}{E^2 - \omega^2 + i\epsilon}. \quad (8.45)$$

This shift corresponds to a shift  $q(t) \rightarrow x(t) = q(t) + F(t)$  for  $F(t)$  the Fourier transform of the second term on the RHS above. This shift by a function does not change the path integral measure since it corresponds in the discretization to shifts by (different) constant amounts for each ordinary integral. So  $Dq = Dx$  and we have

$$\langle 0|0 \rangle_f = \exp \left[ \frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] \quad (8.46)$$

$$\times \int Dx \exp \left[ \frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \tilde{x}(E)(E^2 - \omega^2 + i\epsilon)\tilde{x}(-E) \right]. \quad (8.47)$$

Notice that if we set  $f = 0$ , we just get the second line of the above. We normalize this norm of the ground state to 1. That means for  $f \neq 0$  we get just the first line above:

$$\langle 0|0 \rangle_f = \exp \left[ \frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] \quad (8.48)$$

## The slick way

The above was a bit painful. In the end it's telling us that there is a continuum generalization of the formulas in Appendix D. Here is the slick way to do it using this fact from the outset. It will seem like dark arts at first but you will get used to it.

The path integral we want to compute is

$$\int Dq \exp \left[ \int_{-\infty}^{\infty} dt \left( \frac{\dot{q}^2}{2} - \frac{\omega^2}{2} q^2 + fq \right) \right] \quad (8.49)$$

where we consider the purely imaginary time  $t \rightarrow it$ . We integrate the first term by parts to write this as

$$\int Dq \exp \left[ \int_{-\infty}^{\infty} dt \left( -\frac{1}{2} q (\partial_t^2 + \omega^2) q + fq \right) \right] \quad (8.50)$$

Defining the differential operator  $A = \partial_t^2 + \omega^2$  means that our path integral becomes

$$\int Dq \exp \left[ \int_{-\infty}^{\infty} dt \left( -\frac{1}{2} q A q + fq \right) \right] = \exp \left( \frac{1}{2} \int dt dt' f(t) A^{-1}(t, t') f(t') \right) \quad (8.51)$$

where the answer was obtained as the continuum generalization of (D.14). (The two indices in (D.14) means we should have expected to obtain two times; see next section for some technical details.) We have discussed in Appendix C that the inverse of a differential operator is determined by the Green's function:

$$(\partial_t^2 + \omega^2) A^{-1}(t, t') = \delta(t - t') \implies A^{-1}(t, t') = \int \frac{dE}{-E^2 + \omega^2} e^{iE(t-t')}. \quad (8.52)$$

We see again the shorthand from (B.11) that in Fourier space  $\partial_t^2 \rightarrow -E^2$ . Fourier transforming  $f(t)$  and  $f(t')$  as well gives

$$\exp \left( \frac{1}{2} \int dt dt' \frac{dE}{2\pi} \frac{dE'}{2\pi} \frac{dE''}{2\pi} \frac{\tilde{f}(E') e^{-iE't} \tilde{f}(E'') e^{-iE''t'}}{-E^2 + \omega^2} e^{iE(t-t')} \right) \quad (8.53)$$

Performing the  $t$  and  $t'$  integrals converts the phases into  $(2\pi)^2 \delta(E - E') \delta(E + E'')$ , after which performing the  $E'$  and  $E''$  integrals sets  $E = E' = -E''$ . Up to the overall factor of  $i$  and the factors of  $-i\epsilon$  in the denominator, this gives (8.48). The factor of  $-i\epsilon$  easily follows if we restore the factors of  $\epsilon$  in (8.49). And the overall factor of  $i$  simply follows from doing the Gaussian phase version of this argument instead of the real Gaussian one we did above.

## A discretization (feel free to skip)

Before discretizing we write (8.49) as

$$\int Dq \exp \left[ - \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \delta(t-t') \left( \frac{\dot{q}(t)\dot{q}(t')}{2} - \frac{\omega^2}{2} q(t)q(t') + f(t)q(t') \right) \right] \quad (8.54)$$

Discretizing this means we convert the two integrals in the exponential into two sums and the path integral into a multidimensional integral:

$$\int_{-\infty}^{\infty} d^n q \exp \left[ -\delta t \delta t' \left( \frac{1}{2} \dot{q}^i \delta_{ij} \dot{q}^j - \frac{\omega^2}{2} q^i \delta_{ij} q^j + f^i \delta_{ij} q^j \right) \right]. \quad (8.55)$$

The time  $t$  became the index  $i$ , the time  $t'$  became the index  $j$ , and the continuous Dirac delta function  $\delta(t-t')$  became the discrete Kronecker delta function  $\delta_{ij}$ . Defining  $q^i - q^{i-1} = \dot{q}^i \delta t$ ,  $q^j - q^{j-1} = \dot{q}^j \delta t'$  as before gives

$$\int_{-\infty}^{\infty} d^n q \exp \left[ -\frac{1}{2} (q^i - q^{i-1}) \delta_{ij} (q^j - q^{j-1}) - \frac{\omega^2}{2} q^i \delta_{ij} q^j \delta t \delta t' + f^i \delta_{ij} q^j \delta t \delta t' \right] \quad (8.56)$$

for  $i = 1, \dots, n$ . Now we can write this as a multidimensional Gaussian integral in the form

$$\int_{-\infty}^{\infty} d^n q \exp \left[ -\frac{1}{2} q^i A_{ij} q^j + f_i q^i \right] \quad (8.57)$$

which will allow us to use (D.14). To accommodate the differences  $q^i - q^{i-1}$  and  $q^j - q^{j-1}$  just means we need an  $A_{ij}$  with components on the diagonals immediately above and below the main diagonal, i.e. nonzero components  $a_{i,i-1}$  and  $a_{i,i+1}$ .

## Correlation functions

Fourier transforming from energy  $E$  to time  $t$  lets us write  $\langle 0|0 \rangle_f$  in (8.48) as

$$\langle 0|0 \rangle_f = \exp \left[ \frac{i}{2} \int_{-\infty}^{\infty} dt dt' f(t) G(t-t') f(t') \right] \quad (8.58)$$

where

$$G(t-t') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2 + \omega^2 - i\epsilon} \quad (8.59)$$

is a Green's function for the equation of motion

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t - t') = \delta(t - t'). \quad (8.60)$$

To see this, plug (8.59) into (8.60) and take  $\epsilon \rightarrow 0$ . Notice that the appearance of the  $-i\epsilon$  in the denominator means this is Feynman's Green's function aka the Feynman propagator, which obeys particular boundary conditions. We will write it as  $G$  instead of  $G_F$  from now on since we will only ever be concerned with this Green's function.

We can evaluate (8.59) by closing the contour and using the residue theorem. The poles are at  $E = \pm\omega \mp i\epsilon/(2\omega)$ . For  $t > t'$  we close the contour below and pick up the pole  $E = \omega - i\epsilon/(2\omega)$ , while for  $t < t'$  we close the contour above and pick up the pole  $E = -\omega + i\epsilon/(2\omega)$ . The first contour is negatively oriented (clockwise) and gives  $-2\pi i$  times the residue at the pole, while the second contour is positively oriented (counterclockwise) and gives  $2\pi i$  times the residue. Both cases can be packaged into

$$G(t - t') = \frac{i}{2\omega} \exp(-i\omega|t - t'|). \quad (8.61)$$

We can calculate ground-state correlation functions by using (8.35) with (8.58). A functional derivative acting on  $\langle 0|0\rangle_f$  brings down a Green's function convolved with a source,<sup>18</sup> e.g.

$$\langle 0|T[q(t_1)q(t_2)]|0\rangle = \frac{1}{i} \frac{\delta}{\delta f(t_1)} \frac{1}{i} \frac{\delta}{\delta f(t_2)} \langle 0|0\rangle_f \Big|_{f=0} \quad (8.62)$$

$$= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \int_{-\infty}^{\infty} G(t_2 - t') f(t') \langle 0|0\rangle_f \Big|_{f=0} \quad (8.63)$$

$$= \frac{1}{i} G(t_2 - t_1). \quad (8.64)$$

Notice that  $\delta/\delta f(t_1)$  acts on  $\langle 0|0\rangle_f$  as well, but that term vanishes when we set  $f = 0$  at the end of the calculation.

We can compute arbitrary  $n$ -point time-ordered correlators this way. If  $n$  is odd, e.g.  $n = 1$ , then there will be a stray  $f$  hanging around which when set to zero will set the entire expression to zero. So these correlators vanish. For even  $2n$  we get all  $\frac{1}{n!} \binom{2n}{2} \binom{2n-2}{2} \cdots \binom{2}{2} = \frac{(2n)!}{n!2^n}$  ways of pairing up the derivatives, e.g.

$$\langle 0|T[q(t_1)q(t_2)q(t_3)q(t_4)]|0\rangle = \frac{1}{i^2} [G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}] \quad (8.65)$$

---

<sup>18</sup>I always feel super smart when I say something is convolved with something else. It just means they are multiplied together and then integrated.

where  $G_{ij} = G(t_i - t_j)$ . More generally we have

$$\text{Wick's theorem:} \quad \langle 0|T[q(t_1)\cdots q(t_{2n})]|0\rangle = \frac{1}{i^n} \sum_{\text{pairings}} G_{i_1 i_2} \cdots G_{i_{2n-1} i_{2n}}. \quad (8.66)$$

- You: I can't believe Wick got a named theorem for this stupid result.
- Me: you're welcome. The path-integral approach we're using makes Wick's theorem pretty trivial. It's nastier using "canonical" methods, see e.g. Section 7.A of [4] or Section 4.3 of [5].

## 9 Path integral for free field theory

### 9.1 A bridge from quantum mechanics to quantum field theory

One of the advantages of the path integral approach is that the similarity between quantum mechanics and quantum field theory is even tighter. The equal-time canonical commutation relations in the Heisenberg picture can be written as

$$[x_a(t), p_b(t)] = i\hbar\delta_{ab} \quad \longleftrightarrow \quad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\hbar\delta^3(\mathbf{x} - \mathbf{x}'). \quad (9.1)$$

The eigenstates of operators are

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle, \quad \hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \quad \longleftrightarrow \quad \hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle, \quad \hat{\pi}(\mathbf{x})|\pi\rangle = \pi(\mathbf{x})|\pi\rangle. \quad (9.2)$$

This lets us generalize single-particle position-space wavefunctions to wavefunctionals of quantum fields

$$\psi(\mathbf{x}, t) = \langle \mathbf{x} | \psi(t) \rangle \quad \longleftrightarrow \quad \Psi[\phi; t] = \langle \phi | \Psi(t) \rangle, \quad (9.3)$$

Both wavefunction and wavefunctional satisfy a Schrödinger equation:

$$i\hbar\partial_t\psi(\mathbf{x}) = \hat{H}\psi(\mathbf{x}) \quad \longleftrightarrow \quad i\hbar\partial_t\Psi[\phi; t] = \hat{H}\Psi[\phi; t], \quad (9.4)$$

where the latter equation is a *functional* equation due to the momentum density being represented in the position basis as  $\pi(\mathbf{x}) = -i\hbar\frac{\delta}{\delta\phi(\mathbf{x})}$ .

The overlap between eigenstates of canonically conjugate operators are

$$\langle \mathbf{p} | \mathbf{x} \rangle = \frac{\exp(-i\mathbf{p} \cdot \mathbf{x})}{(2\pi)^{3/2}} \quad \longleftrightarrow \quad \langle \pi | \phi \rangle = \exp\left(-i \int d^3x \pi(\mathbf{x})\phi(\mathbf{x})\right) \quad (9.5)$$

and the resolution of the identity is modified as

$$1 = \int d^3x |\mathbf{x}\rangle\langle\mathbf{x}| = \int d^3p |\mathbf{p}\rangle\langle\mathbf{p}| \quad \longleftrightarrow \quad 1 = \int \mathcal{D}\pi(\mathbf{x}) |\pi\rangle\langle\pi| = \int \mathcal{D}\phi(\mathbf{x}) |\phi\rangle\langle\phi|. \quad (9.6)$$

We are in the Heisenberg picture, so these eigenstates do not evolve in time. That means that this path integral, if we discretize it, corresponds to a bunch of integrals *at a fixed time*, but over all points in space. That is what the notation  $\mathcal{D}\phi(\mathbf{x})$ , without a dependence on time, is supposed to represent (contrast this with  $\mathcal{D}\phi(\mathbf{x}, t)$  in (9.11) below). For example, if we were in one spatial dimension, we could write  $\mathcal{D}\phi(x) = d\phi_1 d\phi_2 \cdots d\phi_N$  where we discretized

space into a lattice of  $N$  points.

The overlap between different eigenstates of the same operator are modified from

$$\langle \mathbf{x}' | \mathbf{x} \rangle = \int d^3 p \langle \mathbf{x}' | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{x} \rangle = \int \frac{d^3 p}{(2\pi)^3} \exp(-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')) = \delta^3(\mathbf{x} - \mathbf{x}') \quad (9.7)$$

to

$$\langle \phi' | \phi \rangle = \int \mathcal{D}\pi(\mathbf{x}) \langle \phi' | \pi \rangle \langle \pi | \phi \rangle = \int \mathcal{D}\pi(\mathbf{x}) \exp\left(-i \int d^3 x \pi(\mathbf{x}) [\phi(\mathbf{x}) - \phi(\mathbf{x}')]\right) = \delta[\phi - \phi']. \quad (9.8)$$

This is a *functional* Fourier transform with a *functional* Dirac delta distribution, satisfying

$$\int d^3 x f(\mathbf{x}) \delta^3(\mathbf{x} - \mathbf{x}') = f(\mathbf{x}') \quad \longleftrightarrow \quad \int D\phi(\mathbf{x}) F[\phi] \delta[\phi - \phi'] = F[\phi']. \quad (9.9)$$

## 9.2 Deriving the path integral

The path integral did not really illustrate its power in the case of the harmonic oscillator, which IMO is more efficiently calculable with canonical methods. That is because the harmonic oscillator is too simple, and it is rarely beneficial to use fancy techniques on simple systems. But the fancy techniques will begin to show their power as we progress to non-Gaussian theories (as in the the toy integral considered above) and gauge theories.

We consider the scalar field theory (4.4) with  $V(\phi) = \frac{1}{2}m^2\phi^2$ , which gives

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \quad \longleftrightarrow \quad \mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \quad (9.10)$$

where the subscript 0 again refers to the fact that the theory is free, i.e. the path integral is Gaussian. The variable  $q(t)$  from quantum mechanics has become the field  $\phi(\mathbf{x}, t)$ , and the source  $f(t)$  we will call  $J(\mathbf{x}, t)$ . To derive the path integral representation of  $\langle 0|0\rangle_J$ , we follow precisely the same steps as in the quantum-mechanical case, except we use the eigenstates of the field operator  $\hat{\phi}(\mathbf{x})$  instead of the position operator  $\hat{x}$  and eigenstates of the canonically conjugate momentum operator  $\hat{\pi}(\mathbf{x})$  instead of the momentum operator  $\hat{p}$ . Using these complete sets of states we can repeat the derivation from quantum mechanics and land on

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\phi(\mathbf{x}, t) e^{i \int d^4 x [\mathcal{L}_0 + J\phi]}. \quad (9.11)$$

We have again used the slightly complexified time contour  $t \rightarrow (1 - i\epsilon)t$  with  $t \in (-\infty, \infty)$

to project to the vacuum, but we will leave this implicit.<sup>19</sup> Notice that the measure is now  $\mathcal{D}\phi(\mathbf{x}, t)$ , meaning we discretize space *and* time. We will often just write this as  $\mathcal{D}\phi$ . The “path integral” is a “path” in the space of field configurations: at some moment in time we have some crazy wiggly field configuration, at the next some other crazy one (not necessarily related to the first by classical evolution), etc. The natural generalization of our Gaussian path integral formula is

$$Z_0(J) = \exp \left[ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 + m^2 - i\epsilon} \right]. \quad (9.12)$$

Let’s see how we get this. We can evaluate it in exactly the same way as the harmonic oscillator. We Fourier transform

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\phi}(k) \quad (9.13)$$

where  $kx = k_\mu x^\mu = -k_0 t + \mathbf{k} \cdot \mathbf{x}$ . We use this to write  $S_0 = \int d^4x (\mathcal{L}_0 + J\phi)$  as

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ -\tilde{\phi}(k)(k^2 + m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k) \right]. \quad (9.14)$$

We again complete the square by a change of variable

$$\tilde{\chi}(k) = \tilde{\phi}(k) - \frac{\tilde{J}(k)}{k^2 + m^2}. \quad (9.15)$$

The measure remains unchanged,  $\mathcal{D}\phi = \mathcal{D}\chi$ , and the action becomes

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 + m^2} - \tilde{\chi}(k)(k^2 + m^2)\tilde{\chi}(-k) \right]. \quad (9.16)$$

As in the harmonic oscillator, the path integral over the second term in  $S_0$  is simply  $Z_0(0) = \langle 0|0 \rangle_{J=0} = 1$ . This is a choice of normalization of measure so that we get unit-norm eigenstates. Then we reproduce (9.12):

$$Z_0(J) = \exp \left[ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 + m^2 - i\epsilon} \right] \quad (9.17)$$

---

<sup>19</sup>In QFT texts a lot of stuff is left implicit. So whenever you see a path integral, you should ask yourself what the (implicit) boundary conditions are and what you’re integrating over – if you stop to think about it it’s usually clear, but if you don’t you’ll sink fast.

$$= \exp \left[ \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right], \quad (9.18)$$

where we defined the Feynman propagator

$$G(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon}. \quad (9.19)$$

The Feynman propagator is a Green's function for the Klein-Gordon equation

$$(-\partial_x^2 + m^2)G(x-x') = \delta^4(x-x'). \quad (9.20)$$

Evaluating (9.19) with the residue theorem gives

$$G(x-x') = i \int \frac{d^3k}{(2\pi)^3 2\omega} e^{ik \cdot (x-x') - i\omega|t-t'|} \quad (9.21)$$

$$= i\theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega} e^{ik(x-x')} + i\theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik(x-x')} \quad (9.22)$$

We can compute correlators by taking variational derivatives of  $Z_0(J)$ :

$$\langle 0|T[\phi(x_1) \cdots]|0\rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} \cdots Z_0(J) \Big|_{J=0}. \quad (9.23)$$

By using the explicit formula (9.18), we get

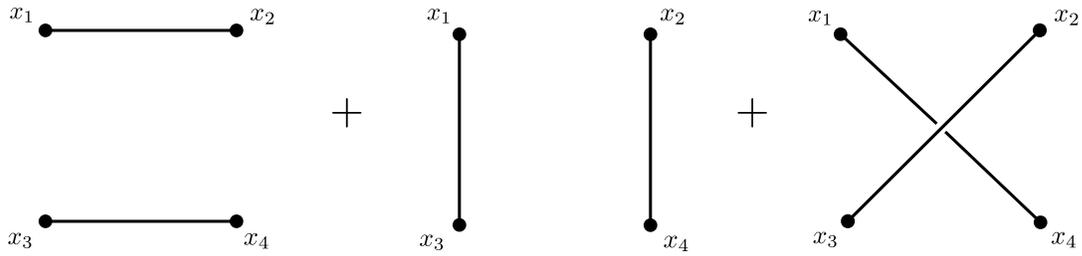
$$\text{Wick's theorem:} \quad \langle 0|T[\phi(x_1) \cdots \phi(x_{2n})]|0\rangle = \frac{1}{i^n} \sum_{\text{pairings}} G_{i_1 i_2} \cdots G_{i_{2n-1} i_{2n}}, \quad (9.24)$$

where  $G_{ij} \equiv G(x_i - x_j)$ . The correlator of an odd number of fields vanishes. For example, for the four-point function we have, via Wick's theorem,

$$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_2)\phi(x_4)]|0\rangle = G(x_1 - x_2)G(x_3 - x_4) + G(x_1 - x_3)G(x_2 - x_4) \quad (9.25)$$

$$+ G(x_1 - x_4)G(x_2 - x_3). \quad (9.26)$$

This is represented in terms of Feynman diagrams, which we will turn to next, as follows:



Notice there are no loop diagrams since the theory is free (see the discussion at the end of Appendix C). We will interpret the diagrams as follows. Particles are created at two spacetime points, and they each propagate to one of the other two spacetime points where they are annihilated. There are three ways this can occur and the total amplitude is the sum of all three possibilities.

## 10 Path integral for interacting field theory: part 1

### 10.1 Feynman diagrams

You may know that much of quantum field theory is concerned with calculating Feynman diagrams. Problems in quantum mechanics can be treated similarly, and with our path integral introduced we can explain what the Feynman diagrams are doing. They are just calculating the (path) integrals introduced above! Say we introduced an anharmonic term into the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2} + \frac{\omega^2}{2}q^2 + \frac{\lambda}{4!}q^4 - fq. \quad (10.1)$$

We can write down the Lagrangian path integral

$$\int Dq \exp \left[ i \int_{-\infty}^{\infty} dt \left( \frac{\dot{q}^2}{2} - \frac{\omega^2}{2}q^2 - \frac{\lambda}{4!}q^4 + fq \right) \right] \quad (10.2)$$

This is not exactly calculable since it is no longer Gaussian. Even the classical equation of motion,

$$\ddot{q} = f - \omega^2q - \frac{\lambda}{3!}q^3, \quad (10.3)$$

obtained by approximating the path integral by saddle point, does not have a simple solution. We can, however, solve the classical equation of motion order-by-order in  $\lambda$ , and encapsulate the contribution at each order in  $\lambda$  by “tree-level” (i.e. classical) Feynman diagrams, which by definition don’t have any loops, as discussed in field theory toward the end of Appendix C.

The full quantum-mechanical problem requires doing the path integral *exactly*, not just by saddle point. We can do this integral perturbatively in  $\lambda$  by expanding the exponential into a series in  $\lambda$  and doing these simpler integrals. The Feynman diagrams are just capturing these integrals! How does this differ from the Feynman diagrams of the process above, which calculates the solution to the classical equation of motion? As discussed in Appendix C, we simply allow lines in Feynman diagrams that close in on themselves when we want to do the full quantum problem.

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*Example 1:* We can illustrate the remarks about Feynman diagrams in an even simpler model,

an ordinary integral. Let's calculate

$$Z = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4 + Jq}. \quad (10.4)$$

We are calling it  $Z$  to conform to notation that will appear later. For  $\lambda = 0$  this is a Gaussian integral we can do as in Appendix D. For  $\lambda \neq 0$  we expand the anharmonic term to get

$$Z = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 + Jq} \left[ 1 - \frac{\lambda}{4!} q^4 + \frac{1}{2} \left( \frac{\lambda}{4!} \right)^2 q^8 + \dots \right]. \quad (10.5)$$

We can now integrate each of these terms separately. For example, we can compute the term of order  $q^{4n}$  by writing it as

$$\frac{1}{n!} \left( -\frac{\lambda}{4!} \right)^n \left( \frac{d}{dJ} \right)^{4n} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 + Jq}. \quad (10.6)$$

This is just derivatives of a Gaussian integral, so we can do it! We are not assured, however, that getting the general- $n$  formula will give us a series that we can then sum to get the full answer. We can, however, write the full answer as

$$Z = \exp \left[ -\frac{\lambda}{4!} \left( \frac{d}{dJ} \right)^4 \right] \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 + Jq} = \sqrt{\frac{2\pi}{\omega}} \exp \left[ -\frac{\lambda}{4!} \left( \frac{d}{dJ} \right)^4 \right] \exp \left[ \frac{J^2}{2\omega^2} \right] \quad (10.7)$$

Calculating to  $O(\lambda^n)$  just requires expanding the exponential on the left to that order and taking the appropriate derivatives. It will be helpful for our future analysis to also expand the  $\exp \left[ \frac{J^2}{2\omega^2} \right]$  term. So we think of  $Z = Z(\lambda, J)$  and compute to a particular order in  $\lambda$  and  $J$ .

As an example, let's calculate the term of order  $\lambda$  and  $J^4$ . This requires expanding the left exponential to order  $\lambda$ , and therefore the right exponential to order  $J^8$ :

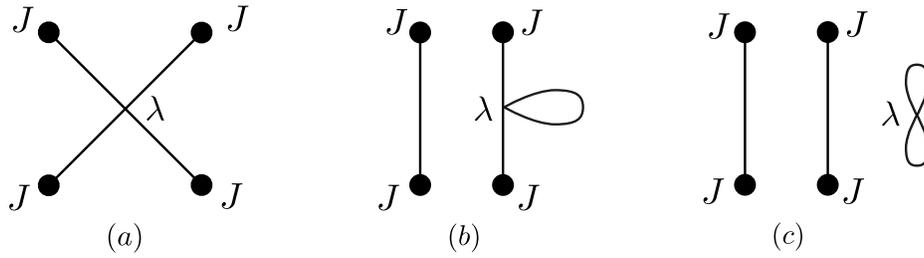
$$-\frac{\lambda}{4!} \left( \frac{d}{dJ} \right)^4 \left[ \frac{1}{4!} \left( \frac{J^2}{2\omega^2} \right)^4 \right] = -\frac{8!}{(4!)^3 (2\omega^2)^4} \lambda J^4, \quad (10.8)$$

where we ignored the prefactor of  $\sqrt{2\pi/\omega}$ . For reasons that will soon become apparent, we can associate a set of Feynman diagrams to any term of order  $\lambda^m$  and  $J^{4n}$ . The rules are

- the diagrams are made of lines and vertices at which four lines meet (four is special since we had the term  $\sim \lambda q^4$ )
- for each vertex we assign a factor of  $(-\lambda)$

- for each line we assign a factor of  $1/\omega^2$
- for each external end assign a factor of  $J$

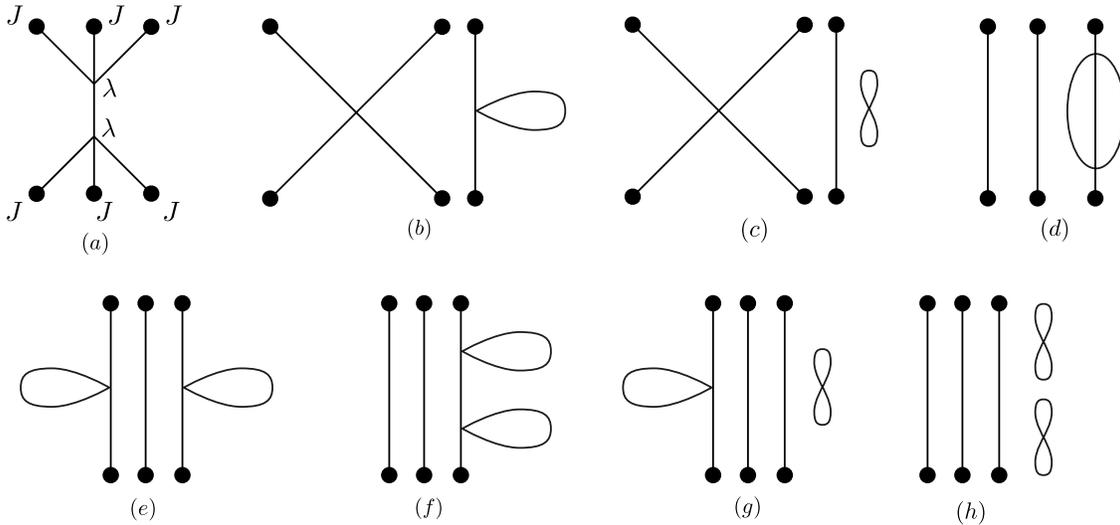
Notice that unlike the rules for the Feynman diagrams for the nonlinear differential equation from Appendix C, here all lines have to either end in a “source”  $J$  or loop back on themselves. If we just accept these rules, then we can draw the following three diagrams for the term of order  $\lambda$  and  $J^4$  we just calculated:



The term of order  $\lambda^2$  and  $J^6$  is

$$\frac{1}{2!} \left( -\frac{\lambda}{4!} \right)^2 \left( \frac{d}{dJ} \right)^8 \left[ \frac{1}{7!} \left( \frac{J^2}{2\omega^2} \right)^7 \right] = \frac{14!}{(4!)^2 6! 7! 2 (2\omega^2)^7} (-\lambda)^2 J^6 \quad (10.9)$$

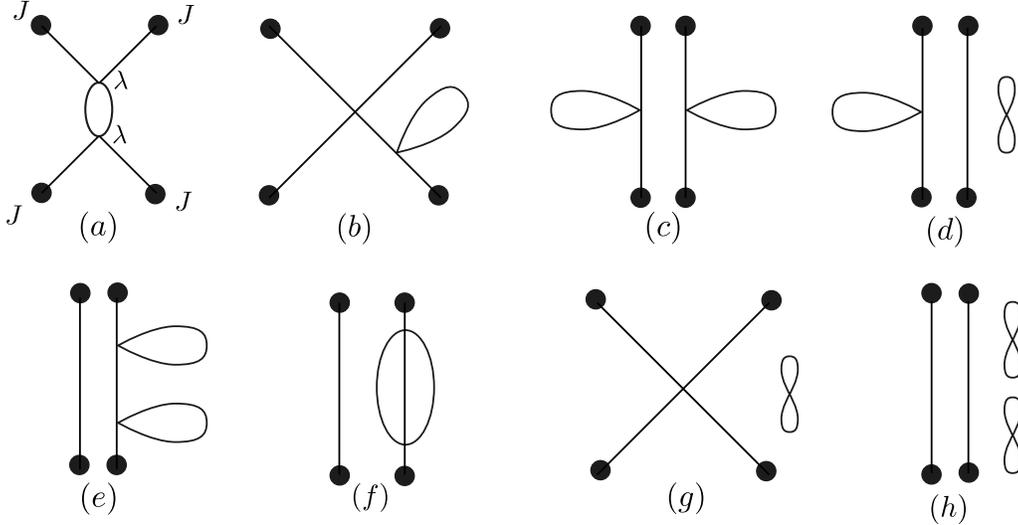
The diagrams we can draw are



The term of order  $\lambda^2$  and  $J^4$  is

$$\frac{1}{2!} \left(-\frac{\lambda}{4!}\right)^2 \left(\frac{d}{dJ}\right)^8 \left[\frac{1}{6!} \left(\frac{J^2}{2\omega^2}\right)^6\right] = \frac{12!}{(4!)^3 3! (2\omega^2)^6} (-\lambda)^2 J^4 \quad (10.10)$$

with diagrams



Finally let's calculate the term of order  $\lambda$  and  $J^0$ :

$$-\frac{\lambda}{4!} \left(\frac{d}{dJ}\right)^4 \left[\frac{1}{2!} \left(\frac{J^2}{2\omega^2}\right)^2\right] = \frac{1}{2!(2\omega^2)^2} (-\lambda) \quad (10.11)$$

and has just one diagram



Notice in each case, there are two distinct sorts of diagrams: ones with loops and ones without. These are called loop diagrams and tree-level diagrams, respectively. **Notice that the term of order  $\lambda^2 J^4$  is all loop diagrams. Given the comments after (10.3), this must mean that if you evaluate the integral by saddle point and expand for small  $\lambda, J$ , there will be no term of order  $\lambda^2 J^4$ . Check this!** The terms of order  $\lambda J^4$  and  $\lambda^2 J^6$  each have one tree-level diagram, diagram (a), and a term of this order indeed appears

in the saddle-point approximation to the integral.

Another way to calculate diagrams is to expand  $Z$  in a series in  $J$  first:

$$Z(J, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4} q^n := Z(0, 0) \sum_{n=0}^{\infty} \frac{1}{n!} J^n G^{(n)}, \quad (10.12)$$

where the coefficients  $G^{(n)}$  are predecessors of the Green's functions we have seen already (and will see again in the field theory context). Now each  $G^{(n)}$  can be expanded in a series in  $\lambda$ .

One more way we can write the middle expression in (10.12) is

$$Z(J, \lambda) = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4} + \sum_{n=1}^{\infty} \frac{1}{n!} J^n \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4} q^n \quad (10.13)$$

$$= Z(J=0, \lambda) + \sum_{n=1}^{\infty} \frac{1}{n!} J^n \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4} q^n \quad (10.14)$$

$$= Z(J=0, \lambda) \left( 1 + Z(J=0, \lambda)^{-1} \sum_{n=1}^{\infty} \frac{1}{n!} J^n \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4} q^n \right) \quad (10.15)$$

$$:= Z(J=0, \lambda) \sum_{N=0}^{\infty} \frac{1}{N!} W(J, \lambda)^N = Z(J=0, \lambda) e^{W(J, \lambda)} \quad (10.16)$$

This will prove useful to distinguish two types of diagrams we can see in the above drawings: connected vs disconnected diagrams. Diagrams (a) at order  $\lambda J^4$ ,  $\lambda^2 J^6$ , and  $\lambda^2 J^4$  are all connected diagrams (we cannot split them into more than one piece without cutting a line), whereas diagrams (c) at those orders are disconnected. In the above representation of the integral, we will see that  $W(J, \lambda)$  is a sum of connected diagrams *only*, whereas  $Z(J, \lambda)$  sums disconnected and connected pieces. There is a close relation to statistical mechanics, where the  $Z$  and  $W$  are related to one another just like the partition function and free energy are related.

---

The example above used Feynman diagrams to evaluate  $Z(J, \lambda)$  order by order. In quantum field theory we will have the generating functional  $Z(J(x), \lambda)$ , and it encodes the correlation functions through derivatives with respect to the source  $J(x)$  as we've seen. Once we have the rules for computing  $Z(J(x), \lambda)$  in perturbation theory, it is relatively straightforward to

write down the rules for computing the correlation functions.

Let's construct the Feynman diagrams for an interacting field theory. We will consider the following path integral:

$$Z(J) = \int D\phi e^{i \int d^4x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\lambda}{3!} \phi^3 + J\phi \right)} \quad (10.17)$$

We already calculated the answer for the free theory,  $\lambda = 0$ , in (9.18):

$$Z_0(J) = \int D\phi e^{iS_0} = \exp \left[ \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right]. \quad (10.18)$$

Just like in (8.38) where we introduced interactions in quantum mechanics, we can decompose this path integral as

$$Z(J) = \int D\phi e^{i \int d^4x \frac{\lambda}{3!} \phi^3} e^{iS_0} = e^{\frac{i\lambda}{3!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3} \int D\phi e^{iS_0} \propto e^{\frac{i\lambda}{3!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3} Z_0(J), \quad (10.19)$$

with  $Z_0(J)$  given by the RHS of (10.18). The proportionality constant appears because the  $i\epsilon$  prescription projects us to something proportional to the ground state, and this proportionality constant is usually absorbed into the path integral measure. This is what happened in the harmonic oscillator and the free scalar, and it led to  $Z_0(0) = 1$ . Since we want  $Z(0) = 1$  in the interacting theory, we are going to have to pick a different measure, and with this measure we have  $\int D\phi e^{iS_0} \propto \exp \left[ \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right]$ . We will soon choose this proportionality constant so that  $Z(0) = 1$ .

By expanding the two exponentials, we can calculate  $Z(J)$  to any order in  $J, \lambda$ . If we want to eventually compute an  $n$ -point function, then we need  $Z(J)$  to order  $J^n$ , so that when we take  $n$  functional derivatives with respect to  $J$  and set  $J = 0$  we get a nonvanishing answer. If we want to calculate to order  $\lambda^m$ , then we need to expand the first exponential in the path integral to order  $m$ , which means we will have  $3m$   $J$  derivatives, so we need to expand  $Z_0(J)$  to order  $J^{n+3m} := J^{2P}$  so that at the end  $Z(J)$  is of order  $J^n$ . We have introduced the parameter  $P$  which will count the number of propagators in a diagram. The parameter  $m := V$  counts the number of vertices. ([1] also defines  $n := E$ , for *external*, since these sources will sit at the edges of Feynman diagrams.) Using these parameters we can write our path integral as

$$Z(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[ \frac{i\lambda}{3!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right]^V \sum_{P=0}^{\infty} \frac{1}{P!} \left[ \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right]^P. \quad (10.20)$$

The  $3V$  sources can act on the  $2P$  sources in many different ways, in particular  $(2P)!/(2P - 3V)!$  different ways. Many of these give the same mathematical expression, but the distinct ones are captured by distinct-looking Feynman diagrams.

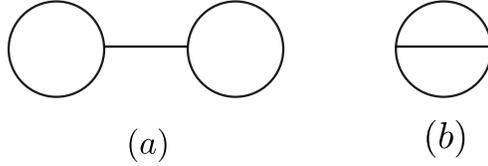
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*Example 2:* To see how this works let's calculate a few cases. First we consider the term of order  $\lambda^2 J^0$ , which is the  $m = 2, n = 0$  case of the above. This means we need to expand  $Z_0(J)$  to order  $J^6$  and the exponential that acts on it to order  $\lambda^2$ :

$$Z(J) \supset \left( \frac{i\lambda}{3!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right)^2 \frac{1}{3!} \left( \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right)^3 \quad (10.21)$$

$$= \frac{-i\lambda^2}{(3!)^3 2^3} \int d^4x_1 \cdots d^4x_8 \left( \frac{\delta}{\delta J_1} \right)^3 \left( \frac{\delta}{\delta J_2} \right)^3 J_3 G_{34} J_4 J_5 G_{56} J_6 J_7 G_{78} J_8, \quad (10.22)$$

where we introduced  $J_i := J(x_i)$  and  $G_{ij} := G(x_i - x_j)$ . Depending on which  $J$ 's the functional derivatives act on, we will get different structures. For example, consider all three  $J_1$ -derivatives acting on  $J_3, J_4, J_5$ , and the  $J_2$  derivatives acting on the rest. This will give a term proportional to  $\int d^4x_1 d^4x_2 G_{11} G_{12} G_{22}$ . Consider instead that the  $J_1$  derivatives act on  $J_3, J_5, J_7$ , and the  $J_2$  derivatives acting on the rest. This will instead give a term proportional to  $\int d^4x d^4x_2 G_{12}^3$ . These correspond to (a) and (b) below, respectively.



The three-point vertices correspond to spacetime points  $x_1$  and  $x_2$ , which need to be integrated over. Notice that the loops in the left diagram correspond to  $G_{11}$ , which begins and ends at  $x_1$ , and similarly  $G_{22}$ . These objects are divergent, since

$$G(x-x) = G(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon}, \quad (10.23)$$

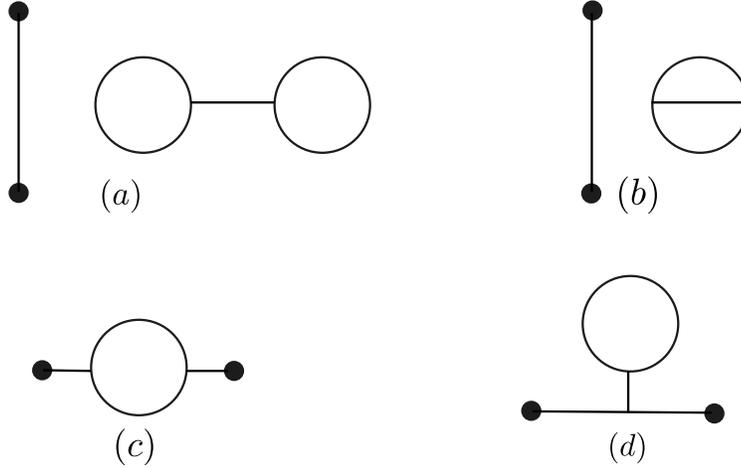
where we lost the  $e^{ik(x-y)}$  phase to close the contour at  $\pm i\infty$  as we usually do. And without the phase we can see that this integral should diverge at large  $k$ . We will postpone dealing with these infinities: it is the topic of renormalization which we will shortly come to. For now we will compute some other diagrams to just see how they work.

Let's consider the term of  $\lambda^2 J^2$ , which is  $m = 2, n = 2$ . That means we need to expand  $Z_0(J)$  to order  $J^8$  and the exponential that acts on it to order  $\lambda^2$ :

$$Z(J) \supset \left( \frac{i\lambda}{3!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right)^2 \frac{1}{4!} \left( \frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right)^4 \quad (10.24)$$

$$= \frac{\lambda^2}{(3!)^2 4! 2^4} \int d^4x_1 \cdots d^4x_{10} \left( \frac{\delta}{\delta J_1} \right)^3 \left( \frac{\delta}{\delta J_2} \right)^3 J_3 G_{34} J_4 \cdots J_9 G_{9,10} J_{10}, \quad (10.25)$$

We can have the following structures. Notice that this is schematically the same as the previous calculation except for the additional  $J_9 G_{9,10} J_{10}$ . So the 6  $J$  derivatives can act on  $J_3$  through  $J_8$  in the two ways as before, resulting in  $\int d^4x_1 d^4x_2 d^4x_9 d^4x_{10} G_{11} G_{12} G_{22} J_9 G_{9,10} J_{10}$  and  $\int d^4x_1 d^4x_2 d^4x_9 d^4x_{10} G_{12}^3 J_9 G_{9,10} J_{10}$ . This gives diagrams (a) and (b) below. We can also have the derivatives act to get  $\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 J_1 G_{12} J_4 G_{34} G_{23}^2$ , represented by diagram (c) below, and  $\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 J_1 J_2 G_{13}^2 G_{23} G_{34} G_{44}$ , represented by diagram (d) below.



For more examples, see Figures 9.1 - 9.11 of [1].

$Z(J)$  at order  $\lambda^2 J^2$  can be used to compute the  $O(\lambda^2)$  correction to the 2-point function (which at  $O(\lambda^0)$  is given just by the free field answer) by taking two functional  $J$  derivatives. But notice some of the terms have this singular  $G_{11}$ , like diagram (d), which corresponds to a loop in the diagram. These led to divergences, but it's not just these pure loops that cause divergences. For example diagram (c) has a  $G_{ij}^2$  in it, which also causes divergences. So we see that these quantum corrections (which always correspond to loops) cause infinities to appear in our calculations.

---

In the example above, there were many more ways we could have acted with the derivatives to give the same diagrams. Let's count the possible ways for a general  $\lambda^V J^E$  diagram. First notice that at  $O(\lambda^V)$ , we have  $(\int d^4x(\delta/\delta J(x))^3)^V$  and that we will have  $V$  vertices. We can rearrange the three  $\delta/\delta J$  factors on each vertex without changing anything; this gives a degeneracy factor of  $(3!)^V$ . But we can also exchange vertices without changing anything, i.e. swapping any  $(\delta/\delta J)^3$  cluster with any other  $(\delta/\delta J)^3$  cluster. This gives a factor of  $V!$ . The sources at the ends of propagators can also be interchanged: for example, if we have a  $G_{12}$  resulting from acting with  $(\delta/\delta J_1)(\delta/\delta J_2)$  on  $\int d^4x_3 d^4x_4 J_3 G_{34} J_4$ , then flipping  $J_3$  and  $J_4$  gives  $G_{21}$ , flipping the propagator. This makes no difference in the final answer due to the symmetry of the Feynman propagator. This factor of 2 is there even if the propagator ends at an external source: the single derivative can act on either  $J$ . This gives a degeneracy factor of  $2^P$  since there are  $P$  propagators. Finally, pairs of sources linked up by a propagator can be interchanged with other pairs of sources linked up by a propagator, giving an additional factor of  $P!$ . Altogether this gives a factor of  $V!(3!)^V P!2^P$  which precisely cancels the prefactor in the term of order  $\lambda^V J^{2P}$  in (10.20)!

The issue, however, is that some of these exchanges we discussed are degenerate with one another and lead to the same pairing of derivatives with sources. So this overcounts the number of terms that give the same results. This is the subject of *symmetry factors*. To understand this, look back at (10.20). One series is a bunch of  $\delta/\delta J$ 's, and the other series is a bunch of  $J$ 's. Each  $\delta/\delta J$  is trying to find a soulmate in one of the  $J$ 's. Turns out any  $J$  will do. The factors we just counted have to do with swapping  $\delta/\delta J$ 's ( $(3!)^V$  and  $V!$  factors above) or swapping  $J$ 's (the  $2^P$  and  $P!$  factors above), to produce distinct pairings of  $\delta/\delta J$ 's with  $J$ 's. But sometimes it will be the case that swapping  $\delta/\delta J$ 's is undone by swapping  $J$ 's! This does not produce a distinct pairing, so multiplying the ways of swapping  $J$  derivatives with the ways of swapping  $J$ 's is an overcounting. We therefore need to divide by an appropriate symmetry factor to account for this. We will often refer to swapping sources linked up by a propagator as reversing the propagator, and interchanging pairs of sources linked up by a propagator with other pairs of sources linked up by a propagator as exchanging propagators.

Let's compute the symmetry factor for diagram (a) at order  $\lambda^2 J^0$  above. While this diagram represents what happens when the derivatives have already acted, it helps to recall the status before the derivatives act, as discussed in the paragraph above. I will therefore refer to each propagator as ending at sources. In diagram (a) each vertex has a pure bubble attached, where by pure bubble I mean a propagator that begins and ends at the same point. Exchanging the sources of this propagator is equivalent to exchanging the appropriate pair

of  $\delta/\delta J$  derivatives at the vertex; this gives a symmetry factor of 2 for each pure bubble. Furthermore, we can exchange the sources of the middle propagator, thereby reversing the propagator, but this is the same as exchanging a  $\delta/\delta J$  from the cluster at one vertex with a  $\delta/\delta J$  from the cluster at another vertex. This is another factor of 2, giving a total symmetry factor of  $2^3$ . Diagram (b) has a symmetry factor of  $2 \times 3!$ : the 2 is from the equivalence between exchanging all the  $\delta/\delta J$  derivatives at one vertex with all the  $\delta/\delta J$  derivatives at the other vertex and reversing all three propagators. The factor of  $3!$  comes from any exchange of the  $3!$  possible exchanges of the three propagators being duplicated by a suitable exchange of derivatives at each vertex.

Notice that some of the diagrams drawn above are *disconnected*, e.g. (a) and (b) of the  $\lambda^2 J^2$  term. Disconnected means what you'd guess – you can separate out subdiagrams by drawing a line between them (that does not intersect the diagram). We can represent the contributions to a general diagram  $D$  in terms of its *connected* subdiagrams  $C_I$ , i.e. subdiagrams that cannot be further divided in the way stated above. The set of all connected diagrams is countable and so  $I$  indexes the nonnegative integers. We have

$$D_{\{n_I\}} = \frac{1}{S_D} \prod_{I=1}^{\infty} (C_I)^{n_I} . \quad (10.26)$$

The integer  $n_I$  counts the number of  $C_I$ 's in  $D_{\{n_I\}}$ , and  $S_D$  is an *additional* symmetry factor for the full diagram  $D_{\{n_I\}}$ . This symmetry factor has to do with symmetries between  $C_I$ 's; the symmetry of any given connected diagram is already included in  $C_I$ . This new symmetry factor is just  $S_D = \prod_{I=1}^{\infty} n_I!$ . The  $n_I!$  is from the permutations of the  $n_I$  equivalent diagrams  $C_I$ , and the product simply counts all different equivalence classes. Our partition function sums over *all possible* diagrams  $D_{\{n_I\}}$ , which can be written as a sum over all possible strings of nonnegative integers  $\{n_I\}$

$$Z(J) \propto \sum_{\{n_I\}} D_{\{n_I\}} = \sum_{\{n_I\}} \prod_{I=1}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} = \prod_{I=1}^{\infty} \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} = \exp \left( \sum_I C_I \right) \quad (10.27)$$

The switch of the sum with the product is a bit tricky. To convince yourself, pick a few terms you expect to be in the sum of all possible diagrams, and show that it is reproduced in the form where the product comes before the sum. Another consistency check that assures all the diagrams are there (checking the numerical factor requires a bit more work) is to notice that  $\prod_{I=1}^{\infty} \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I}$  contains all possible strings  $C_1^{n_1} C_2^{n_2} \dots$ . So we see that if we want  $Z(0) = 1$ , we can simply leave out the vacuum diagrams, i.e. those with no sources

$J$ .<sup>20</sup> With this normalization we can write

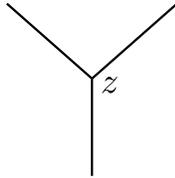
$$Z(J) = \exp [iW(J)] = \exp \left[ \sum_{I \neq \{0\}} C_I \right], \quad (10.28)$$

where the notation  $I \neq \{0\}$  means we leave out the vacuum diagrams from the sum and so  $W(0) = 0$ .

Now that we understand how to construct these diagrams from the equations, we can simply write down the Feynman rules to draw pictures and read off the corresponding equations. We will do this for the position-space time-ordered correlation functions. These are obtained by functional derivatives of  $Z(J)$ , after which we set  $J = 0$ . For example, to compute  $\langle T[\phi(x)\phi(y)] \rangle$  to order  $\lambda^n$  we simply take our calculation of  $Z(J)$  to order  $\lambda^n$  and calculate  $(1/i)\delta/\delta J(x)(1/i)\delta/\delta J(y)Z(J)\Big|_{J=0}$ . All this does is convert the external sources into operators  $\phi(x)$  and  $\phi(y)$  at fixed spacetime points (i.e. not to be integrated over). If we look at (10.20), this means that every  $iJGJ$  term will have both sources paired with  $(1/i)\delta/\delta J$  terms, so every propagator will contribute a  $-iG$ . Vertices contribute  $i\lambda \int d^4z$ , while external points don't contribute anything (since we already accounted for the  $-i$  due to conversion from source to operator insertion in our analysis of the propagators). The Feynman rules are therefore as follows:

$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3$ : position-space Feynman rules for time-ordered correlators

- For each propagator  $x \text{ --- } y = -iG(x-y),$



- For each vertex  $= i\lambda \int d^4z$

<sup>20</sup>Without elaboration, it seems like a lucky coincidence that leaving out diagrams gives us the normalization we want. After all, why is it justified to ignore diagrams? The idea is that the free theory vacuum  $|0\rangle_{\text{free}}$  gets upgraded to the interacting theory vacuum  $|0\rangle_{\text{interacting}}$  through a summation of all of these vacuum diagrams! They are interpreted as a renormalization of the vacuum. So we are not ignoring these diagrams, they are already included when considering  $|0\rangle_{\text{interacting}}$ , so we do not need to include them again. This is consistent with the fact that this prescription gives  $Z(0) = \text{interacting}\langle 0|0\rangle_{\text{interacting}} = 1$ , i.e. the interacting vacuum should have unit norm like all states.

- For each external point  $x$   = 1
- Divide by the symmetry factor
- Sum up all fully connected diagrams

That's it! We can use these rules to calculate, for example, the  $\lambda^2$  correction to the two-point function  $\langle T[\phi(x)\phi(y)] \rangle$ . The diagrams are precisely (a) - (d) in the  $\lambda^2 J^2$  calculation of Example 2, except now the external sources are fixed operator insertions  $\phi(x)\phi(y)$ . Using our Feynman rules, diagram (c) is equal to

$$-\frac{\lambda^2}{4} \int d^4w d^4z G(x-w)G(w-z)^2G(z-y) \quad (10.29)$$

### Path integrals and probability theory

Let's pursue our analogy between path integrals and probability theory. We already saw that  $Z$  is the generating functional of correlation functions, in analogy to the moment-generating function in (8.27). Recall the moment-generating function  $Z(s)$  computes the moments  $\langle x^n \rangle = (d/ds)^n Z(s)|_{s=0}$ . Now we have this object  $W = -i \log Z$ . In probability theory, the logarithm of the moment-generating function is the **cumulant-generating function**. These cumulants are not just  $\langle x^n \rangle$ , but instead they add or subtract "disconnected" pieces that can be written as products of expectation values  $\langle x^{a_1} \rangle^{b_1} \dots \langle x^{a_m} \rangle^{b_m}$  with  $a_1 b_1 + \dots + a_m b_m = n$ .

## 10.2 Infinities and renormalization

As we saw above for calculations of  $Z(J)$ , if we use these Feynman rules to calculate correlation functions, we will see infinities all over the place. This is inevitable when we treat an interacting QFT in perturbation theory: you find that e.g. the  $O(\lambda^2)$  correction to  $\langle \phi(x)\phi(y) \rangle$  in the theory in the theory  $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda\phi^3$  is divergent. This is BAD. A perturbative correction should be small, not big, let alone infinite! This happens in all sorts of quantum field theory calculations: corrections to atomic energy levels, particle masses, and more. It was a major problem in the historical development of QFT.

The basic resolution is that we are focusing on a quantity that isn't measurable. All measurable quantities will actually come out finite. But computationally it will be simpler to still compute non-measurable infinite quantities, but to regulate them in a certain way. Then at the end when we put the pieces together to get the answer for a measurable quantity, the regulator will drop out of the answer.

As an example of this philosophy, let's consider the Casimir effect. We start with the Hamiltonian for a free scalar field, which we repeat here:

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) + \frac{1}{2} (2\pi)^{-3} \int d^3k \omega_{\mathbf{k}} V \quad (10.30)$$

where  $V$  is the volume of space. In this case we do not (yet) allow ourselves the constant shift  $\Omega_0$  from before. The vacuum energy is therefore

$$E = \langle 0|H|0\rangle = V \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} = \frac{V}{4\pi^2} \int k^3 dk = \infty. \quad (10.31)$$

The vacuum seems to have infinite energy! But we know that the vacuum energy is not observable: it is only differences in energy that are meaningful (as you may be familiar with from the notion of potential energy).

Let's set up a calculation where differences in this infinite vacuum energy are calculable and finite. This is due to Casimir. Consider the theory in one spatial dimension, in a finite-sized box of size  $a$ . If the energy depends on  $a$ , then there will be a force on the walls of the box due to  $F = -dE/da$ . But we need to calculate the energy delicately, since it is infinite:

$$E(a) = \sum_{n=1}^{\infty} \frac{\omega_n}{2} = \frac{\pi}{2a} \sum_{n=1}^{\infty} n \quad (10.32)$$

where the frequencies are discrete  $\omega_n = \pi n/a$  since we are in a box with boundary conditions requiring the field to vanish at the endpoints. This sum is clearly infinite, as is its derivative  $-dE/da$ . We regulate it with a "heat kernel regulator":

$$E(a) = \frac{\pi}{2a} \sum_{n=1}^{\infty} n e^{-\frac{n}{\Lambda a}} = \frac{\pi}{2a} \sum_{n=1}^{\infty} n e^{-\epsilon n} \quad (10.33)$$

The regulator  $\Lambda$  makes sure that frequencies  $\omega_n \gg \Lambda$  are exponentially suppressed in the sum. We defined  $\epsilon = (\Lambda a)^{-1}$  to simplify calculation:

$$\sum_{n=1}^{\infty} n e^{-\epsilon n} = -\partial_{\epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} = -\partial_{\epsilon} \frac{1}{e^{\epsilon} - 1} = \frac{e^{-\epsilon}}{(1 - e^{-\epsilon})^2} = \frac{1}{\epsilon^2} - \frac{1}{12} + \frac{\epsilon^2}{240} + \dots \quad (10.34)$$

Thus the energy becomes

$$E(a) = \frac{\pi a}{2} \Lambda^2 - \frac{\pi}{24a} + \dots, \quad (10.35)$$

The full energy, however, is a contribution from inside the box of size  $a$  and from outside the box. The outside is infinite, but let's just say it has size  $L$  which we will take to infinity at the end. Then the total energy is

$$E_{tot}(a) = E(a) + E(L - a) = \frac{\pi L}{2} \Lambda^2 - \frac{\pi}{24} \left( \frac{1}{L - a} + \frac{1}{a} \right) \quad (10.36)$$

and the force is

$$F(a) = -\frac{dE_{tot}}{da} = \frac{\pi}{24} \left( \frac{1}{(L - a)^2} - \frac{1}{a^2} \right) + \dots \quad (10.37)$$

Taking the limit  $L \rightarrow \infty$  and restoring units gives us a force

$$F(a) = -\frac{\pi \hbar c}{24a^2}. \quad (10.38)$$

Notice it is a purely quantum-mechanical effect, being proportional to  $\hbar$ .

### 10.2.1 Regulator independence

This was just one way of doing the calculation, which amounted to  $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots = -\frac{1}{12}$ . That seems crazy! But in fact, this force has been measured (in the case of the electromagnetic field), and it is in precise agreement with this prediction. The substitution  $1 + 2 + 3 + \dots = -\frac{1}{12}$  is also behind the claim that superstring theory lives in 10 dimensions. But how do we know that other ways of regulating the sum won't give you different answers? Well, this particular case is very well-studied. Many other regulators give you the same answer, e.g.

$$E(a) = \frac{1}{2} \sum_n \omega_n \theta(\pi\Lambda - \omega_n) \quad (\text{hard cutoff}) \quad (10.39)$$

$$E(a) = \frac{1}{2} \sum_n \omega_n e^{-\left(\frac{\omega_n}{\pi\Lambda}\right)^2} \quad (\text{Gaussian}) \quad (10.40)$$

$$E(a) = \frac{1}{2} \sum_n \omega_n \left( \frac{\omega_n}{\mu} \right)^{-s} \quad (\zeta\text{-function}) \quad (10.41)$$

In fact one can argue that a wide class of regulators should give you the same answer, as indeed Casimir did in his original paper (see Section 15.3 of [4] for more).

But why does this work, physically? The basic idea is that we are calculating something that is an infrared effect, i.e. it does not depend on the very high-energy modes in the system. This is often the case we will be in, and the reasoning may be somewhat different

in various cases, but let's think about the physical setup here. In the lab, one would set up electromagnetic waves in a box, and the discrete frequencies would be due to the boundary conditions of the electromagnetic field at the walls of the box. But a box is made of atoms, and the discretization will be due to the interaction of the electromagnetic waves and the atoms of the box. But if we go to sufficiently high frequency the electromagnetic waves won't interact with the atoms and won't be discretized according to the above. They won't even see the box! So these should not really be entering our sum. Thus, any sane regulator which eliminates the high-frequency modes will give the same answer. Because it is an infrared effect, we are usually safe with any regulator that doesn't mess with the infrared too much (even if it is violent in the ultraviolet).

In quantum field theory this will keep happening: very high frequency modes will lead to divergences that need to be removed. The way we will do them is through introduction of “counterterms,” which are unobservable, infinite terms we can add to the Lagrangian to cancel against other infinities which appear in intermediate steps. For example, in this case we can recall our old friend  $\Omega_0$ , which we added to the Lagrangian in (5.1) for exactly this reason! How we pick counterterms depends on the regulator chosen. For example, for the 1d free scalar with heat kernel regulator just treated we would pick

$$\Omega_0 = -\frac{\pi}{2}\Lambda^2, \tag{10.42}$$

which when integrated against space gives an energy  $-\pi\Lambda^2/(2a)$  which cancels the infinite contribution to the energy  $+\pi\Lambda^2/(2a)$ .

### 10.2.2 Lattice regulator

A natural way to regulate *any* infinity from quantum field theory is by discretizing space, i.e. turning three-dimensional space into a lattice of  $N \times N \times N$  points. This will turn the QFT into  $N^3$  harmonic oscillators living at each lattice point, coupled through the kinetic term. (For gauge theories the story is a little more complicated.) Notice that the infrared is insensitive to the existence of a lattice: for example, space in our world can be discretized on a very tiny scale and we wouldn't be able to tell by doing infrared experiments. So this is again something that messes with the ultraviolet. Now the vacuum energy is clearly finite, since there are a finite number of interacting quantum-mechanical systems. A way to see this explicitly is that since we discretized space, the frequencies are going to be in a finite range. This is because high frequencies that would have many oscillations between lattice points are meaningless, since the lattice can't see those oscillations! This discretization of space leading to a finite frequency range is the “inverse” of the fact that when we put space

in a finite range the frequencies became discretized. Mathematically, our field  $\phi(x)$  becomes, in 1 space dimension,  $N$  values  $\phi_j$ ,  $j = 0, \dots, N - 1$ . This is connected to frequency space by a discrete Fourier transform,

$$\phi_j = \sum_{n=0}^{N-1} \phi_n e^{\frac{2\pi i n}{N} j}, \quad (10.43)$$

so we see that the frequencies now have a maximum. While a lattice regulator is conceptually beautiful, and helps organize one's thinking, it is often not very practical. To see how to get the right Casimir energy out of this regulator, see Section 15.2 of [4] – it's a lot more painful than the heat kernel regulator we used above.

### 10.3 Returning to correlators

The Casimir energy discussion above was a digressive analogy to the pesky infinities that appear due to loops in Feynman diagrams. We learned in this analogy that we should always speak in terms of physically measurable quantities (like the change in energy or force on the walls of the box). A computationally helpful tool was to introduce some sort of regulator. The final answer could then be expressed in terms of a divergent piece, a finite piece, and a vanishing piece. The divergent piece diverges as we take the regulator away ( $\Lambda \rightarrow \infty$  in our example), the finite piece is independent of  $\Lambda$ , and the vanishing piece goes to zero as  $\Lambda \rightarrow \infty$ . We saw that we could introduce a counterterm to cancel the divergent piece, which let us work with pleasantly finite quantities everywhere. In our analysis of Feynman diagrams, we saw that the momentum integrals led to infinities. Thus we will want to regulate the very high momentum modes. Our final answer will have divergent pieces, and we will need counterterms to make sure we get finite answers.

Let's return to our  $\phi^3$  theory

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\lambda}{3!} \phi^3. \quad (10.44)$$

The couplings in any physical theory should correspond to things we measure. For example Newton's constant is determined by measuring gravitational attraction. Similarly, the couplings in the above Lagrangian should be finite quantities corresponding to something measurable. Time-ordered correlation functions are measurable, so a natural guess is that  $\lambda_0$  corresponds to the proportionality constant in the three-point function  $\langle 0|T[\phi_0(x_1)\phi_0(x_2)\phi_0(x_3)]|0\rangle$ . The problem is that the three-point function is only proportional to  $\lambda_0$  if we work at tree level or in the semiclassical approximation. Once we draw loop diagrams, there are corrections.

We will have something like

$$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle = \lambda(\cdots) + \lambda^3(\cdots) + \dots \quad (10.45)$$

We can define a *renormalized* coupling  $\lambda_r$  to equal the value of the three-point function at a particular set of spacetime points  $x_i$ . This correlator is finite, and so  $\lambda_r$  will be finite. We will see that  $\lambda_r$  is some infinite series in terms of  $\lambda$  and that  $\lambda$  is actually infinite (more precisely, once we introduce a regulator,  $\lambda$  will depend on the regulator in a way that diverges as we take the regulator away)! So it is a pretty crummy parameter to stick into a Lagrangian. Working in this way, in terms of physical parameters defined by some experimental conditions, actually leads us to replace the Lagrangian (10.44) with

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_r^2}{2}\phi_r^2 + \frac{\lambda_r}{3!}\phi_r^3 - \frac{\delta_Z}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 + \frac{\delta_\lambda}{3!}\phi_r^3 + Y\phi_r, \quad (10.46)$$

with the three  $\delta_i$ 's,  $Y$ ,  $m_r$ , and  $\lambda_r$  determined by the normalization of our field and various experimental conditions, e.g. such-and-such correlator at such-and-such spacetime points behaves like such-and-such. For example, one of the conditions we will require is  $\langle 0|\phi_r(x)|0\rangle = 0$ , which may require a constant shift in our field (to subtract off whatever existing one-point function there is), which is why we have the  $Y\phi_r$  term in the Lagrangian above. Sometimes authors use  $Z_i := 1 + \delta_i$ , e.g.  $Z_m = 1 + \delta_m$  (for the field renormalization the notation is instead  $Z_\phi = 1 + \delta_Z$ ); this is the notation [1] sticks to, although at the end of Chapter 9 he also defines  $A = Z_\phi - 1$ ,  $B = Z_m - 1$ .  $A$  and  $B$  are simply our  $\delta_Z$  and  $\delta_m/m^2$ .

While this looks different than our Lagrangian (10.44), notice that it just corresponds to a bunch of redefinitions, e.g.  $\phi = \phi_r\sqrt{Z_\phi} + \text{const.} = \phi_r\sqrt{1 + \delta_Z} + \text{const.}$ ,  $\lambda(1 + \delta_Z)^{3/2} = \lambda_r + \delta_\lambda$ , etc. So we see that counterterms are not ad-hoc things we add to our Lagrangian solely to cancel infinities. Instead, we simply split up the “bare” parameters  $m_0$  and  $\lambda_0$  into renormalized parameters  $m_r$  and  $\phi_r$  (and shift and “renormalize” our bare field  $\phi$  to  $\phi_r$ ), and what we are left with are the counterterms. Of course, we will need to be careful to choose the  $\delta_i$ 's so that we are sure that we maintain our experimental definitions of the renormalized couplings  $m_r$ ,  $\lambda_r$ . We can treat these counterterms just as in (10.20): we simply trade  $\phi_r$  for  $-i\delta/\delta J$  to write these counterterms as functional differential operators acting on  $Z_0(J)$ . We will see this in more detail soon.

To study this in more detail it helps to turn finally to studying scattering cross-sections. These are the natural things seen at collider experiments, but they are also nice because the amplitudes for S-matrices are field-redefinition-invariant. We need a formula that relates correlation functions in position space, which is what we know how to calculate, to scattering

cross-sections. This is the LSZ formula, which we now turn to.

## 11 LSZ formula

We want a formula for scattering amplitudes in a general interacting theory, something that can tell us e.g. if we smash together two protons at very high energy, what is the probability that a Higgs boson appears in the final state? Our discussion below will follow Chapter 5 of [1]. For a more detailed elaboration see Chapters 13.5 - 14.1 of [3] and references therein.

Recall our free field expression

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left( a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx} \right) \quad (11.1)$$

and our expression for the creation operator

$$a^\dagger(\mathbf{k}) = \int d^3x e^{ikx} (-i\partial_t \phi(x) + \omega_{\mathbf{k}}\phi(x)) = -i \int d^3x e^{ikx} \overleftrightarrow{\partial}_t \phi(x), \quad (11.2)$$

where  $f \overleftrightarrow{\partial} g = f\partial g - (\partial f)g$ . In the free theory, we want to define an operator that creates a particle localized in momentum space near  $\mathbf{k}_1$  and in position space near the origin:

$$a_1^\dagger = \int d^3k f_1(\mathbf{k}) a^\dagger(\mathbf{k}), \quad f_1(\mathbf{k}) \propto \exp\left(-\frac{(\mathbf{k} - \mathbf{k}_1)^2}{4\sigma^2}\right). \quad (11.3)$$

This is a wave packet with width  $\sigma$  in momentum space. [1] calls this a time-independent operator, but he means in the Schrödinger sense (it has the usual Heisenberg time-dependence). We can create the state  $a_1^\dagger a_2^\dagger |0\rangle$  and evolve in the Schrödinger picture to  $t = -\infty$ . As long as  $\mathbf{k}_1 \neq \mathbf{k}_2$ , these wave packets will head in different directions as we rewind time and become well-separated. This seems to provide a good “in” state, i.e. a state of two separated particles that we want to crash together.

What about the interacting theory? In this case we define analogously

$$a^\dagger(\mathbf{k}) = -i \int d^3x e^{ikx} \overleftrightarrow{\partial}_t \phi_r(x), \quad a_1^\dagger = \int d^3k f_1(\mathbf{k}) a^\dagger(\mathbf{k}), \quad (11.4)$$

where they are now time-dependent, even as Schrödinger operators. The field  $\phi_r$  is related to the field in the Lagrangian  $\phi$  in a way we now make precise. To keep the normalizations similar to the free theory, we need to shift and rescale (or “renormalize”) our field  $\phi$ . In the

free theory we have

$$\text{free theory:} \quad \langle 0|\phi(x)|0\rangle = 0, \quad \langle k|\phi(x)|0\rangle = e^{-ikx} \quad (11.5)$$

where both expressions follow immediately from insertion of (11.1). In the interacting theory we define our renormalized field  $\phi_r$  so that

$$\text{interacting theory:} \quad \langle 0|\phi_r(x)|0\rangle = 0, \quad \langle k|\phi_r(x)|0\rangle = e^{-ikx}. \quad (11.6)$$

The vacuum  $|0\rangle$  is the vacuum of the full interacting theory (in many texts this is denoted  $|\Omega\rangle$ ), and  $|k\rangle$  is a one-particle state with 4-momentum  $k$  in this theory, with  $k^2 = -m_{\text{ph}}^2$  where the label ph stands for physical, since this corresponds to the physical mass of the particle. We can assure these normalizations beginning with  $\phi$  in the interacting theory as follows. Let's say there is a nonzero one-point function  $\langle 0|\phi(x)|0\rangle = \langle 0|e^{-iPx}\phi(0)e^{iPx}|0\rangle = \langle 0|\phi(0)|0\rangle \neq 0$ , where we used  $P^\mu|0\rangle = 0$  to obtain an  $x$ -independent one-point function. We can then define a new field  $\tilde{\phi}$  with vanishing one-point function as  $\tilde{\phi}(x) = \phi(x) - \langle 0|\phi(0)|0\rangle$ . Furthermore, we generally have

$$\langle k|\tilde{\phi}(x)|0\rangle = \langle k|e^{-iPx}\tilde{\phi}(0)e^{iPx}|0\rangle = e^{-ikx}\langle k|\tilde{\phi}(0)|0\rangle = e^{-ikx}\sqrt{Z}, \quad (11.7)$$

where we defined the Lorentz-invariant quantity<sup>21</sup>  $\langle k|\tilde{\phi}(0)|0\rangle \equiv \sqrt{Z}$ , the capital Z and square root simply being a convention. So we can rescale  $\tilde{\phi}$  to obtain a renormalized field

$$\phi_r(x) = Z^{-1/2}\tilde{\phi}(x) = Z^{-1/2}(\phi(x) - \langle 0|\phi(0)|0\rangle) \quad (11.8)$$

which satisfies (11.6). We will now assume that we can use  $a_i^\dagger$  from (11.3),<sup>22</sup> except now written in terms of the renormalized field  $\phi_r$ , to create an initial state of two particles as

$$|i\rangle = \lim_{t \rightarrow -\infty} a_1^\dagger(t)a_2^\dagger(t)|0\rangle. \quad (11.9)$$

By an appropriate choice of normalization of  $f_1$  we can obtain  $\langle i|i\rangle = 1$ , which we will assume. Had we not used the renormalized field then we would just stick in a factor of  $Z^{-1}$ .

<sup>21</sup>To see that this is Lorentz invariant, stick in Lorentz transformations  $U(\Lambda)$  as  $\langle k|\tilde{\phi}(0)|0\rangle = \langle k|U^{-1}U\tilde{\phi}(0)U^{-1}U|0\rangle = \langle \Lambda k|\tilde{\phi}|0\rangle$ ; the dependence on  $\Lambda$  must disappear, which means this should be a function of  $k^2$ , a Lorentz-invariant quantity.

<sup>22</sup>For an argument that multiparticle states created by  $a_i^\dagger$  can be ignored in the limits  $t \rightarrow \pm\infty$ ; the basic point is that these states have higher energy which leads to more suppression due to oscillatory phases, see equations (5.19) - (5.23) of [1].

We can also create a final state by acting with operators at  $t = +\infty$ :

$$|f\rangle = \lim_{t \rightarrow +\infty} a_1^\dagger a_2^\dagger |0\rangle. \quad (11.10)$$

We could have put more (or fewer) particles in our final state. The amplitude we want is

$$\langle f|i\rangle = \langle 0|T[a_{1'}(+\infty)a_{2'}(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)]|0\rangle, \quad (11.11)$$

where we stuck in a time-ordering symbol, which we are allowed to do since everything is already time-ordered.

We want to express this amplitude in terms of the field  $\phi_r$ . We first derive an equation relating  $a_1(t = +\infty)$  to  $a_1(t = -\infty)$ :

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt \partial_t a_1^\dagger(t) \quad (11.12)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x \partial_t \left( e^{ikx} \overleftrightarrow{\partial}_t \phi_r(x) \right) \quad (11.13)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (\partial_t^2 + \omega_{\mathbf{k}}^2) \phi_r(x) \quad (11.14)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (\partial_t^2 + \mathbf{k}^2 + m_{\text{ph}}^2) \phi_r(x) \quad (11.15)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (\partial_t^2 - \overleftarrow{\nabla}^2 + m_{\text{ph}}^2) \phi_r(x) \quad (11.16)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (\partial_t^2 - \overrightarrow{\nabla}^2 + m_{\text{ph}}^2) \phi_r(x) \quad (11.17)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial^2 + m_{\text{ph}}^2) \phi_r(x) \quad (11.18)$$

$$(11.19)$$

The first equality is the FTC, the second substituted the definition of  $a_1^\dagger$  (11.3), and in the second-to-last we used the fact that the wavepacket only has Gaussian support so does not contribute boundary terms in the integration by parts. (To see this first perform the  $d^3k$  integral; the Fourier transform of a Gaussian is another Gaussian, so we obtain a Gaussian in position space which gives us the falloff we need.)

Now if we were in the free scalar field theory, this last expression would be zero since the field operator  $\phi_r$  would obey the Klein-Gordon equation. This is consistent with the fact that the field operator  $a_1^\dagger$  is time-independent, so the difference we are constructing should vanish. But in an interacting theory it is nonzero, and the Klein-Gordon equation is not

true. So we have the formula

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial^2 + m_{\text{ph}}^2) \phi_r(x) \quad (11.20)$$

and its Hermitian conjugate

$$a_1(+\infty) = a_1(-\infty) + i \int d^3k f_1(\mathbf{k}) \int d^4x e^{-ikx} (-\partial^2 + m_{\text{ph}}^2) \phi_r(x). \quad (11.21)$$

This is pretty sweet, because we can now plug these into (11.11) and use the time-ordering symbol to move every  $a(-\infty)$  to the right and annihilate the vacuum ket; we also move every  $a^\dagger(+\infty)$  to the left and annihilate the vacuum bra. We can also take  $\sigma \rightarrow 0$  and choose a normalization such that  $f_1(\mathbf{k}) = \delta^{(3)}(\mathbf{k} - \mathbf{k}_1)$ . The full answer, generalizing to  $n$  in-particles and  $n'$  out-particles, becomes

$$\langle f|i \rangle = i^{n+n'} \int d^4x_1 e^{ik_1x_1} (-\partial_1^2 + m_{\text{ph}}^2) \cdots d^4x'_1 e^{-ik'_1x'_1} (-\partial_1'^2 + m_{\text{ph}}^2) \cdots \langle 0|T[\phi_r(x_1) \cdots \phi_r(x'_1) \cdots]|0 \rangle \quad (11.22)$$

We can get a bit more intuition for this by Fourier transforming to momentum space:

$$\prod_{i=1}^n \frac{i}{k_i^2 + m_{\text{ph}}^2} \prod_{i=1}^{n'} \frac{i}{k_{i'}^2 + m_{\text{ph}}^2} \langle f|i \rangle = \langle 0|\phi_r(k_1) \cdots \phi_r(k_n) \phi_r(k_{1'}) \cdots \phi_r(k_{n'})|0 \rangle. \quad (11.23)$$

The right-hand-side is understood to be the Fourier Transform of the time-ordered correlation function. So if we want the transition amplitude  $\langle f|i \rangle$ , we Fourier transform the time-ordered correlator and put the momenta close to their on-shell values  $k_i^2 \sim m^2$ ; we will see an  $(n + n')$ 'th order pole develop if all momenta are placed on-shell, and the coefficient of that pole is the transition amplitude.<sup>23</sup>

**Key takeaways:** LSZ in our renormalization scheme requires the conditions  $\langle 0|0 \rangle = 1$ ,  $\langle 0|\phi_r(x)|0 \rangle = 0$ ,  $\langle k|\phi_r(x)|0 \rangle = e^{-ikx}$ , and implies a further condition  $\langle k, n|\phi_r(x)|0 \rangle = 0$  where  $|k, n \rangle$  is a multiparticle state with total 4-momentum  $k$  and  $n$  is a label for all other characteristics e.g. relative momenta.  $|0 \rangle$  is the vacuum of the full, interacting theory. An

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<sup>23</sup>If following along with [5], notice the field is normalized differently such that the propagators have a factor of  $Z$  upstairs; they pick our normalization for  $\phi_r$  only much later, in Section 10.2, thereby eliminating these pesky factors of  $Z$ . This choice, where the field  $\phi_r$  which satisfies  $\langle k|\phi_r(x)|0 \rangle = e^{-ikx}$  is *also* the field that appears in the Lagrangian, is part of the *on-shell renormalization scheme* we are developing. This is not necessary; the field in the Lagrangian can instead satisfy  $\langle k|\phi(x)|0 \rangle = Ze^{-ikx}$ , for example in the *minimal subtraction scheme* (see e.g. Ch. 27 of [1] or Section 18.3.1 of [4]). In such a situation the LSZ formula has to be modified by multiplying the RHS by a factor of  $Z^{-1/2}$  for each external particle. By a similar token, the mass parameter in the Lagrangian  $m_r$  will not necessarily correspond to the physical mass  $m_{\text{ph}}$ . But in the on-shell renormalization scheme they are one and the same, as we will soon see.

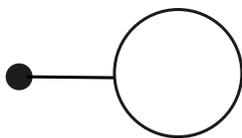
implication of the above is that our asymptotic states, created by  $a^\dagger$ , are single-particle momentum eigenstates in the full, interacting theory. They are on-shell,  $k^2 = -m_{\text{ph}}^2$ . The asymptotic states will correspond to external Feynman lines, which must therefore be exact momentum eigenstates. While these are difficult to calculate, since they have to take into account all loop corrections to the external lines, we will see in Section 15 that they have an isolated pole at the on-shell value  $k^2 = -m_{\text{ph}}^2$ , with residue  $Z$ . The location of this pole is sometimes called the pole mass, and our scheme says that the Lagrangian mass parameter  $m_r$  satisfies  $m_r = m_{\text{ph}}$ . Our scheme also says  $Z = 1$ . Anyway, this means we can just use the propagator  $1/(k^2 + m_r^2 - i\epsilon)$  for these external lines, and **not take into account all the loops on external lines**. As long as we are going on-shell for the external lines, all these loops do is shift the bare mass  $m$  to the pole or physical mass  $m_{\text{ph}}$ . Ignoring these loops on external lines goes by the name of amputation, and diagrams without such loops are called amputated Feynman diagrams.

## 12 Scattering amplitudes

We return to our study of general interacting scalar field theory. But before getting into scattering amplitudes let's revisit (10.46) and understand the role of the counterterms a little better. We'll look at the  $Y\phi$  counterterm in particular. Notice that if we had the action (10.44) without any counterterms, we would calculate

$$\langle 0|\phi_0(x)|0\rangle = \frac{1}{i} \frac{\delta}{\delta J(x)} Z(J) \Big|_{J=0} = \frac{\delta}{\delta J(x)} W(J) \Big|_{J=0} \quad (12.1)$$

This quantity is nonzero, and has all sorts of contributions. They are all the diagrams that contribute to  $Z(J)$  at order  $J$ , i.e. all diagrams with a single source  $J$ , except since we are computing a correlation function we remove the source  $J$ . Since we are omitting all vacuum diagrams to maintain  $Z(0) = 1$ , this excludes any disconnected vacuum diagrams. At leading order we have the diagram



Using our Feynman rules we wrote down at the end of Section 10.1, we have a factor of  $-iG(x-y)$  for the propagator beginning at  $x$  and ending at the three-point vertex which we will call  $y$ , another propagator  $-iG(y-y)$  for the loop, an integral  $i\lambda \int d^4y$  for the three-point vertex, and we divide by 2 for the symmetry factor of the loop:

$$\langle 0|\phi_0(x)|0\rangle = \frac{1}{2} i\lambda \int d^4y \frac{1}{i} G(x-y) \frac{1}{i} G(y-y) + O(\lambda^3) \quad (12.2)$$

So this screws up one of the conditions we needed for the validity of the LSZ formula. But that's where the  $Y\phi_r$  counterterm comes in, singing “Here I come to save the day!”. Remember that this appeared due to a shift to cancel the one-point function, so we should see how the final theory with counterterms (10.46) exhibits a vanishing one-point function. It does this because  $Y\phi_r$  leads to an additional vertex that can be drawn, with its own Feynman rule. This vertex is one where a single line can simply end, and the Feynman rule assigns this vertex a factor of  $iY \int d^4z$  if the spacetime point of the vertex is  $z$ . We will assume that  $Y = O(\lambda)$ , which we will see shortly is self-consistent. The relevant diagram is then



and contributes  $iY \int d^4y \frac{1}{i} G(x-y)$ . The one-point function becomes

$$\langle 0|\phi_r(x)|0\rangle = (iY + \frac{1}{2}(i\lambda)\frac{1}{i}G(0)) \int d^4y \frac{1}{i} G(x-y) + O(\lambda^3) \quad (12.3)$$

For this to vanish at  $O(\lambda)$ , we pick

$$Y = \frac{1}{2}i\lambda G(0) + O(\lambda^3). \quad (12.4)$$

This framework is only sensible if  $Y$  is real (because  $\phi$  is Hermitian and we need our Hamiltonian, which has the term  $Y\phi$  in it, to be Hermitian). To check this we need to evaluate

$$G(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon}, \quad (12.5)$$

which we have already discussed is infinite. Armed with some notion of regularizing these sorts of infinities, we know that we need to introduce some sort of ultraviolet cutoff  $\Lambda$  on the high-frequency modes. We will do this in a way that preserves the Lorentz transformation properties of  $G(x-y)$ :

$$G(x-y) \longrightarrow \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \left( \frac{\Lambda^2}{k^2 + \Lambda^2 - i\epsilon} \right)^2. \quad (12.6)$$

This integral is now convergent (we will see how to evaluate it later), and for  $\Lambda \gg m$  we have

$$G(0) = \frac{i}{16\pi^2} \Lambda^2. \quad (12.7)$$

Thus we see that  $Y$  is real, as required, and infinite, as we warned counterterms would be. The *physical* quantity,  $\langle 0|\phi_r(x)|0\rangle$ , is zero, at least to linear order in  $\lambda$  which is how far we have calculated. We could continue this to higher orders in  $\lambda$  although it gets more complicated. But in any event this is generally how working with counterterms will go: when calculating physically measurable quantities the non-counterterm part of the action will produce some infinities that will be cancelled by the counterterm part of the action, leaving the physical, finite part.

Once we have tuned  $Y$  to ensure  $\langle 0|\phi_r(x)|0\rangle = 0$  to all orders in  $\lambda$ , then the sum of

all connected diagrams with a single source is zero. These are known as tadpole diagrams. Furthermore, a wonderful simplification occurs for all of our Feynman diagrams. It turns out **we can ignore any diagram that you can split into two pieces by cutting across a single line such that one of the two diagrams has no sources.** The sourceless subdiagram can simply be thought of as a tadpole subdiagram, so regardless of what it is attached to, it will be cancelled by the  $Y\phi_r$  counterterm. So we will exclude any diagrams with tadpole subdiagrams.

We have learned to calculate  $Z(J) = e^{iW(J)}$ , with the normalization  $W(0) = 0$  assuring  $Z(0) = \langle 0|0\rangle = 1$  as needed. In free theory we have

$$\frac{1}{i}G(x_1 - x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle, \quad (12.8)$$

where we will from now on drop the subscript  $F$  for Feynman although that is the Green's function we will be using. In the interacting theory we will define

$$\frac{1}{i}G^{\text{exact}}(x_1 - x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \quad (12.9)$$

$G^{\text{exact}}$  is not a Green's function even though it has the letter  $G$ . It is just the full time-ordered correlation function in the interacting theory, i.e. with all corrections in whatever couplings  $\lambda_i$  we may have. We define the shorthand

$$\delta_j = \frac{1}{i} \frac{\delta}{\delta J(x_j)}. \quad (12.10)$$

The two-point correlator can be written as

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \delta_1\delta_2 Z(J)|_{J=0} = e^{iW(J)} (\delta_1\delta_2 iW(J) - (\delta_1 iW(J))(\delta_2 iW(J))) \Big|_{J=0} \quad (12.11)$$

$$= \delta_1\delta_2 iW(J)|_{J=0} \quad (12.12)$$

The four-point correlator can be written as

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \delta_1\delta_2\delta_3\delta_4 Z(J) \quad (12.13)$$

$$= [\delta_1\delta_2\delta_3\delta_4 iW + (\delta_1\delta_2 iW)(\delta_3\delta_4 iW) + (\delta_1\delta_3 iW)(\delta_2\delta_4 iW) + (\delta_1\delta_4 iW)(\delta_2\delta_3 iW)]_{J=0} \quad (12.14)$$

We now consider the LSZ formula for two ingoing and two outgoing particles:

$$\langle f|i\rangle = i^4 \int d^4x_1 d^4x_2 d^4x'_1 d^4x'_2 e^{i(k_1x_1+k_2x_2-k'_1x'_1-k'_2x'_2)} \quad (12.15)$$

$$\times (-\partial_1^2 + m^2)(-\partial_2^2 + m^2)(-\partial_{1'}^2 + m^2)(-\partial_{2'}^2 + m^2) \quad (12.16)$$

$$\times \langle 0|T\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|0\rangle. \quad (12.17)$$

Let's say we input the contribution of the term  $(\delta_1\delta_{1'}iW)(\delta_2\delta_{2'}iW) = \frac{1}{i}G^{\text{exact}}(x_1-x'_1)\frac{1}{i}G^{\text{exact}}(x_2-x'_2)$  from the four-point function into the LSZ formula. Then we get a contribution to  $\langle f|i\rangle$  that looks like

$$- \int d^4x_1 d^4x_2 d^4x'_1 d^4x'_2 e^{i(k_1x_1+k_2x_2-k'_1x'_1-k'_2x'_2)} F(x_{11'})F(x_{22'}) \quad (12.18)$$

where  $F(x_{ij}) = (-\partial_i^2 + m^2)(-\partial_j^2 + m^2)G^{\text{exact}}(x_{ij})$  and  $x_{ij'} = x_i - x_{j'}$ . To do the integrals we write

$$\exp[i(k_ix_i - k'_ix'_i)] = \exp\left[\frac{i}{2}((k_i + k'_i)(x_i - x'_i) + (k_i - k'_i)(x_i + x'_i))\right] \quad (12.19)$$

and switch to variables  $x_{ij'} = x_i - x_{j'}$  and  $x_{ij}^+ = (x_i + x_{j'})/2$ . The integrals over  $x_{ij}^+$  give delta functions and the integrals over  $x_{ii'}$  implement a Fourier transforms, resulting in

$$-(2\pi)^4\delta^4(k_1 - k'_1)(2\pi)^4\delta^4(k_2 - k'_2)\tilde{F}(\bar{k}_{11'})\tilde{F}(\bar{k}_{22'}), \quad (12.20)$$

where  $\tilde{F}(k)$  is the Fourier transform of  $F$  and  $\bar{k}_{ij'} = (k_i + k_{j'})/2$ . These delta functions tell us that the momentum of incoming particle 1 (2) is identical to the momentum of outgoing particle 1' (2'), so particles 1 and 2 never scattered off each other! This is a “disconnected” contribution to the four-point function. The term  $(\delta_1\delta_{2'}iW)(\delta_2\delta_{1'}iW)$  behaves the same way. The last disconnected term,  $(\delta_1\delta_2iW)(\delta_{1'}\delta_{2'}iW)$  is a bit different, since the momentum delta functions enforce zero momentum for the incoming state and zero momentum for the outgoing state. But the incoming (and outgoing) state cannot have a vanishing momentum four-vector as long as it has some energy (which it does if we have massive particles):  $k_1^0 + k_2^0 = 2\gamma m > 0$ . The Feynman diagrams for these disconnected contributions are like the ones at the end of Chapter 9, although in this case we have the exact propagators not the free-field Green's functions.

From here on out we will consider only the fully connected contributions. This is just a choice of what to feed into the LSZ machine. These arise via functional derivatives of  $W$ . So

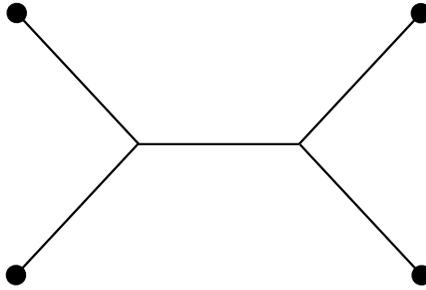
we define the connected correlation functions by

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle_c = \delta_1\cdots\delta_n iW(J)\Big|_{J=0}, \quad (12.21)$$

and we plug this into the LSZ formula to ensure we only get fully connected scattering amplitudes. For our four-point function, we then have

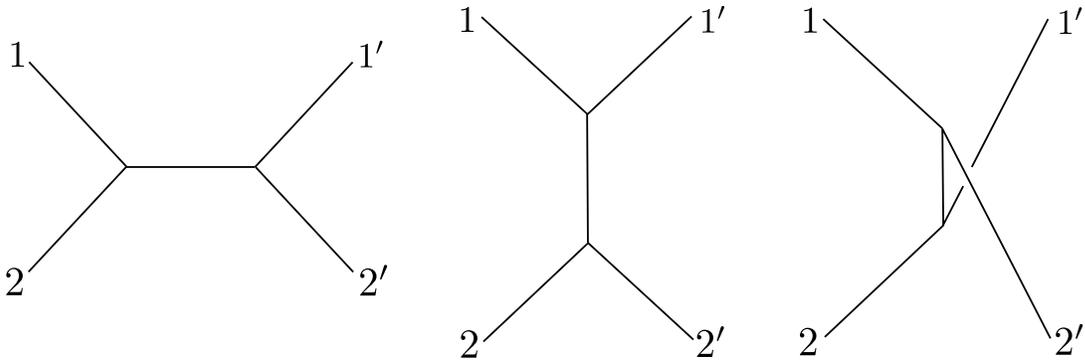
$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|0\rangle_c = \delta_1\delta_2\delta_{1'}\delta_{2'} iW\Big|_{J=0}. \quad (12.22)$$

The diagram that contributes to  $W$  at lowest order in  $\lambda$  is given by

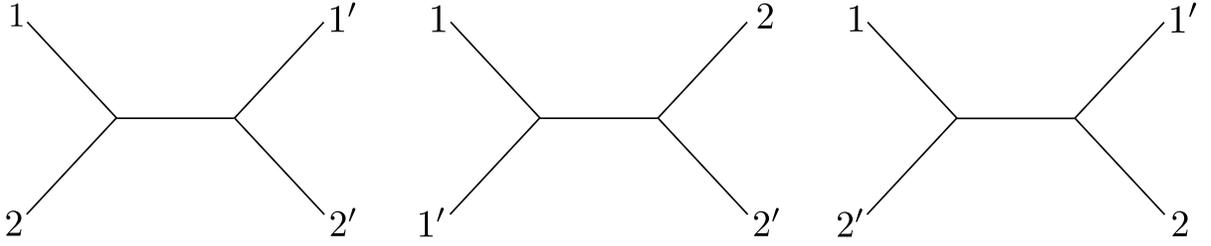


This has a symmetry factor of 8 due to (a) factor of 2 from interchanging the  $(\delta/\delta J)$  cluster at one vertex with the cluster at another vertex, while simultaneously swapping the external propagators attached to them and reversing the internal propagator, (b) two factors of 2, one each from swapping two  $(\delta/\delta J)$  factors in a cluster that go into the external propagators, and simultaneously swapping the propagators.

We have to act on this diagram with four  $\delta$ 's, which gives us  $4!$  possible ways of pairing the  $\delta$ 's with the  $J$ 's. There are three distinct pairings, with a degeneracy of eight per pairing, which cancels the symmetry factor of 8 from above. This gives the diagrams



Notice the cancellation of the symmetry factor makes sense from the perspective that the external points have unique identifiers now, so the interchanges from before are no longer symmetries. Looking at the diagrams, we see that the three unique pairings correspond to whether 1 gets paired with 2, 1', or 2' at its vertex. It is unconventional because one likes to keep the ingoing and outgoing particles in their same location and then change up the structure of the Feynman diagrams, although if we flouted this convention we could draw all three Feynman diagrams as below:



But nobody does this, so we will not do it either.

The lack of symmetry factors is unique to tree-level diagrams. We can now use our Feynman rules from the end of Section 10.1 to get

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|0\rangle_c = (i\lambda)^2(-i)^5 \int d^4y d^4z G_{yz} \quad (12.23)$$

$$\times \left( G_{x_1y}G_{x_2y}G_{x'_1z}G_{x'_2z} + G_{x_1y}G_{x'_1y}G_{x_2z}G_{x'_2z} + G_{x_1y}G_{x'_2y}G_{x_2z}G_{x'_1z} \right) + O(\lambda^4), \quad (12.24)$$

where  $G_{ij} := G(i - j)$ . This is what we want to plug into the LSZ formula. We get

$$(-\partial_i^2 + m^2)G(x_i - y) = \delta^4(x_i - y) \quad (12.25)$$

whenever a Klein-Gordon operator acts on a Green's function, which LSZ assures us will always happen for every external leg. This allows us to do the  $x_1, x_2, x'_1, x'_2$  integrals to get

$$\langle f|i\rangle = i\lambda^2 \int d^4y d^4z G_{yz} \left( e^{i(k_1y+k_2y-k'_1z-k'_2z)} + e^{i(k_1y+k_2z-k'_1y-k'_2z)} + e^{i(k_1y+k_2z-k'_1z-k'_2y)} \right) \quad (12.26)$$

$$+ O(\lambda^4). \quad (12.27)$$

We can plug in our Feynman propagator

$$G_{yz} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(y-z)}}{k^2 + m^2 - i\epsilon} \quad (12.28)$$

to write the integrand as a bunch of phase factors which can be integrated to obtain:

$$\langle f|i\rangle = i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2\pi)^8}{k^2 + m^2 - i\epsilon} [\delta^4(k_1 + k_2 - k)\delta^4(k'_1 + k'_2 + k) \quad (12.29)$$

$$+ \delta^4(k_1 - k'_1 + k)\delta^4(k'_2 - k_2 + k) + \delta^4(k_1 - k'_2 + k)\delta^4(k'_1 - k_2 + k)] + O(\lambda^4) \quad (12.30)$$

$$= i\lambda^2(2\pi)^4 \delta^4(k_1 + k_2 - k'_1 - k'_2) \quad (12.31)$$

$$\times \left[ \frac{1}{(k_1 + k_2)^2 + m^2 - i\epsilon} + \frac{1}{(k_1 - k'_1)^2 + m^2 - i\epsilon} + \frac{1}{(k_1 - k'_2)^2 + m^2 - i\epsilon} \right] + O(\lambda^4). \quad (12.32)$$

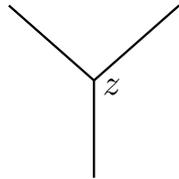
**What a beauty!** That was some hard work, but boy is the final answer a marvel. The overall delta function tells us that momentum is conserved, which is a good sanity check. Stripping off this factor and a constant lets us define a scattering matrix element  $\mathcal{T}$ :

$$\langle f|i\rangle = (2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}}) i\mathcal{T}. \quad (12.33)$$

In many QFT references  $\mathcal{T}$  is defined by subtracting the identity matrix from the S-matrix  $\mathcal{S}$ , but here we have effectively done this by sticking to fully connected graphs only. By meditating on the example above and playing with a couple more, we can write down a set of Feynman rules to compute  $i\mathcal{T}$ :

$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3$ : position-space Feynman rules for the S-matrix  $i\mathcal{T}$

- For each propagator  $x \text{---} y = -iG(x - y),$



- For each vertex  $= i\lambda \int d^4z$

- For each external line  $x \bullet \text{---} = e^{\pm ikx}$ : + is incoming, - is outgoing

- Divide by the symmetry factor
- Sum up all fully connected, amputated diagrams

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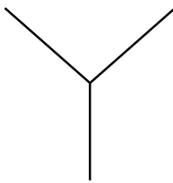
It will help to write these rules down in momentum space, since that is often how we will evaluate them. It will help to put arrows on our propagators. For  $n$  incoming and  $n'$  outgoing we will start with  $n$  external lines with arrows pointing “into” the diagram, i.e. away from the external points, and the  $n'$  external lines will have arrows pointing “away” from the diagram, i.e. toward the external points. We then draw all possible fully connected graphs filling in these external legs, where each internal leg gets assigned its own arbitrary momentum. Notice that in (12.29) - (12.30) we had momentum-conserving delta functions for each vertex in the graph; this is a general rule. We then have the following Feynman rules:

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$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3$ : momentum-space Feynman rules for the S-matrix  $i\mathcal{T}$

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• For each propagator  =  $\frac{-i}{k^2+m^2-i\epsilon}$

• For each vertex  =  $i\lambda$

• For each external line  = 1

- Impose momentum conservation at each vertex
  - Momentum running in loops will not be fixed; integrate over these undetermined momenta  $\int \frac{d^4p}{(2\pi)^4}$ .
  - Divide by the symmetry factor
  - Sum up all fully connected, amputated diagrams
-

These sets of rules have not taken into account the counterterms we have discussed. So if you apply them you will see divergences from e.g. the loop momentum integrals. Let's now bring in the counterterms, writing the theory in the form (10.46), which we reproduce here:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_r^2}{2}\phi_r^2 + \frac{\lambda_r}{3!}\phi_r^3 - \frac{\delta_Z}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{\delta_m}{2}\phi_r^2 + \frac{\delta_\lambda}{3!}\phi_r^3 + Y\phi_r, \quad (12.34)$$

$$= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}} \quad (12.35)$$

We already saw that the  $Y\phi_r$  counterterm let us focus on diagrams without tadpole subdiagrams. That is its role in life. We can treat these counterterms just as in (10.20): we simply trade  $\phi_r$  for  $-i\delta/\delta J$  to write these counterterms as functional differential operators acting on  $Z_0(J)$ :

$$Z(J) \propto e^{i\int d^4x \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} e^{i\int d^4x \mathcal{L}_{\text{ct}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J). \quad (12.36)$$

By the usual arguments then we see that the term  $(\delta_\lambda/3!)\phi_r^3$  introduces a new type of three-point vertex, and the term  $(-\delta_Z/2)\partial_\mu\phi_r\partial^\mu\phi_r - (\delta_m/2)\phi_r^2$  introduces a new kind of two-point vertex, as it can be written

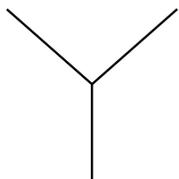
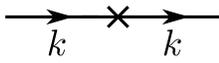
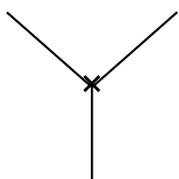
$$\frac{i}{2} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) (\delta_Z \partial^2 - \delta_m) \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \quad (12.37)$$

by integrating by parts. Notice that this is really a vertex, not some sort of new propagator, e.g. you can have many of these vertices sprinkled along a single ordinary propagator, and you will pick up as many vertex factors as you have two-point vertices. In momentum space the  $\partial^2$  becomes  $-k^2$ , so in the end we have the following Feynman rules. **For simplicity from here on out we will now drop all the  $r$  subscripts, but make sure you know what theory and what counterterms are being considered in such situations!**

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3 - \frac{\delta_Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\delta_m}{2}\phi^2 + \frac{\delta_\lambda}{3!}\phi^3 + Y\phi:$$

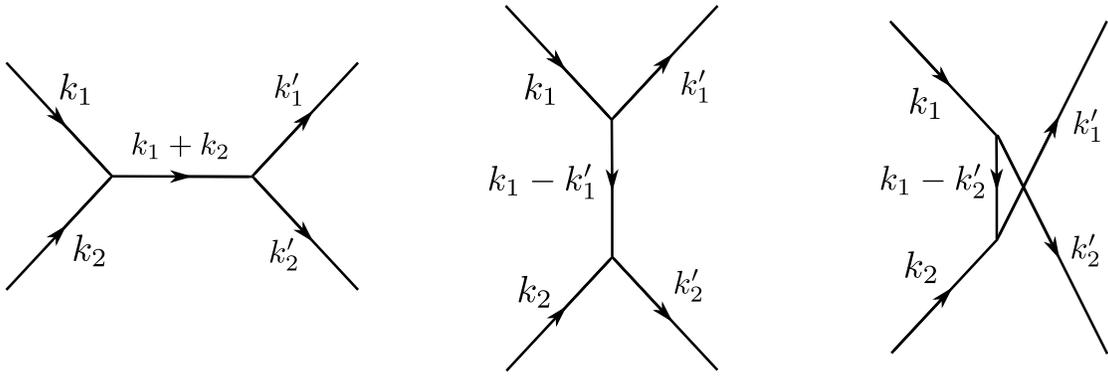
momentum-space Feynman rules for the S-matrix  $i\mathcal{T}$

• For each propagator  $\xrightarrow{k}$   $= \frac{-i}{k^2+m^2-i\epsilon}$

- For each vertex  =  $i\lambda$
- For each external line  = 1
- For each vertex  =  $-i(k^2\delta_Z + \delta_m) \sim k^2O(\lambda^2) + O(\lambda^2)$
- For each vertex  =  $i\delta_\lambda \sim O(\lambda^2)$

- Impose momentum conservation at each vertex
- Momentum running in loops will not be fixed; integrate over these undetermined momenta  $\int \frac{d^4p}{(2\pi)^4}$ .
- Divide by the symmetry factor
- Sum up all fully connected diagrams without tadpole subdiagrams to compute  $i\mathcal{T}$  up to any order  $O(\lambda^n)$  of interest.

For example, drawing the  $O(\lambda^2)$  momentum-space Feynman diagrams for 2-to-2 scattering is given by the following:



The counterterms are irrelevant at this order because there is no diagram with four external legs and any number of counterterm vertices that will contribute at  $O(\lambda^2)$ .

### 13 Cross sections and decay rates

Now that we have the scattering amplitude  $\mathcal{T}$ , we want to convert it into something that is measured in an experiment. We will eventually restrict to 2-to-2 scattering in  $\phi^3$  theory, for which we found (12.32) in the previous section, reproduced here:

$$\mathcal{T} = \lambda^2 \left[ \frac{1}{(k_1 + k_2)^2 + m^2} + \frac{1}{(k_1 - k'_1)^2 + m^2} + \frac{1}{(k_1 - k'_2)^2 + m^2} \right] + O(\lambda^4), \quad (13.1)$$

where we dropped the  $-i\epsilon$ 's for notational simplicity.

Before restricting to the case above, let's analyze a general 2-to-2 scattering process.  $k_1$  and  $k_2$  are the incoming momenta while  $k'_1$  and  $k'_2$  are the outgoing momenta. Overall momentum conservation says  $k_1 + k_2 = k'_1 + k'_2$ . All particles are on-shell,  $k_i^2 = -m_i^2$ , where in general we can have distinct masses. Let's analyze this scattering process in the center-of-mass (CM) frame. This means  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$ . We orient  $\mathbf{k}_1$  along the positive z-axis. So the only remaining input parameter to specify is  $|\mathbf{k}_1|$ . We could equivalently specify the total energy  $E_1 + E_2$ . Better yet, let's define a Lorentz scalar  $s := -(k_1 + k_2)^2$ , which in the CM frame reduces to  $(E_1 + E_2)^2$ . The parameter  $s$  is therefore called the CM energy squared. Using  $E_1^2 = \mathbf{k}_1^2 + m_1^2$  and  $E_2^2 = \mathbf{k}_1^2 + m_2^2$  (recall  $\mathbf{k}_2 = -\mathbf{k}_1$ ) lets us solve for  $|\mathbf{k}_1|$  in terms of  $s$ :

$$|\mathbf{k}_1|_{CM} = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2}, \quad \text{CM frame.} \quad (13.2)$$

Now we think of the two outgoing particles. Momentum conservation tells us  $\mathbf{k}'_1 + \mathbf{k}'_2 = \mathbf{0}$ , while energy conservation tells us  $(E'_1 + E'_2)^2 = s$ . Thus we again have

$$|\mathbf{k}'_1|_{CM} = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m_1'^2 + m_2'^2)s + (m_1'^2 - m_2'^2)^2}, \quad \text{CM frame.} \quad (13.3)$$

The only remaining parameter specifying the final state is the angle  $\theta$  between  $\mathbf{k}_1$  and  $\mathbf{k}'_1$ . We will specify it instead through the Lorentz scalar  $t := -(k_1 - k'_1)^2$ , which is related to  $\theta$  by

$$t = -k_1^2 - k_1'^2 + 2k_1 \cdot k'_1 = m_1^2 + m_1'^2 - 2E_1 E'_1 + 2|\mathbf{k}_1| |\mathbf{k}'_1| \cos \theta, \quad \text{any frame} \quad (13.4)$$

where we evaluated the product in an arbitrary reference frame. In the CM frame we simply pick  $\theta = \theta_{CM}$ . The Lorentz scalars  $s$  and  $t$  are two of the three so-called Mandelstam variables, defined as

$$s := -(k_1 + k_2)^2 = -(k'_1 + k'_2)^2, \quad (13.5)$$

$$t := -(k_1 - k'_1)^2 = -(k_2 - k'_2)^2, \quad (13.6)$$

$$u := -(k_1 - k'_2)^2 = -(k_2 - k'_1)^2. \quad (13.7)$$

These are not all independent, there is an overall constraint

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2. \quad (13.8)$$

The utility of these variables can be seen by writing our amplitude from before

$$\mathcal{T} = \lambda^2 \left[ \frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right] + O(\lambda^4) \quad (13.9)$$

We now consider the fixed target (FT) frame or lab frame, where particle 2 is at rest ( $\mathbf{k}_2 = 0$ ) and bombarded by particle 1. In this frame  $s = -(k_1 + k_2)^2 = (E_1 + m_2)^2 - \mathbf{k}_1^2$ . Now we have  $E_1^2 = \mathbf{k}_1^2 + m_1^2$  and  $E_2^2 = m_2^2$ , which we can again use to solve for  $|\mathbf{k}_1|$  in terms of  $s$  and the masses:

$$|\mathbf{k}_1|_{FT} = \frac{1}{2m_2} \sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2}, \quad \text{FT frame.} \quad (13.10)$$

Comparing this with the CM frame (13.2) gives

$$m_2 |\mathbf{k}_1|_{FT} = \sqrt{s} |\mathbf{k}_1|_{CM}. \quad (13.11)$$

We are allowed to equate the two equations even though they are expressing frame-dependent quantities because the RHS's are written in a frame-independent fashion. We will use this later.

Now we come to state our scattering experiment, which we assume to take place in a large box of volume  $V$  and for large time  $T$ . The number of outgoing particles will be arbitrary but we will have only two incoming particles. Our overlap is given by

$$\langle f|i \rangle = (2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}}) i\mathcal{T}. \quad (13.12)$$

To get a probability, we should square this and divide by the norms of the initial and final states:

$$P = \frac{|\langle f|i \rangle|^2}{\langle f|f \rangle \langle i|i \rangle}. \quad (13.13)$$

The numerator of this expression is given by

$$|\langle f|i\rangle|^2 = [(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}})]^2 |\mathcal{T}|^2. \quad (13.14)$$

A delta function squared doesn't usually make sense, but we can write

$$[(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}})]^2 = (2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}}) \times (2\pi)^4 \delta^4(0) \quad (13.15)$$

and interpret the latter term as

$$(2\pi)^4 \delta^4(0) = \int d^4x e^{i0 \cdot x} = VT. \quad (13.16)$$

The norm of a single-particle state is given by  $\langle k|k\rangle = (2\pi)^3 2k^0 \delta^3(\mathbf{0}) = 2k^0 V$ , so we have

$$\langle i|i\rangle = 4E_1 E_2 V^2, \quad \langle f|f\rangle = \prod_{j=1}^{n'} 2k_j^0 V. \quad (13.17)$$

If we divide by the total time  $T$  we get a probability per unit time

$$\dot{P} = \frac{(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}}) V |\mathcal{T}|^2}{4E_1 E_2 V^2 \prod_{j=1}^{n'} 2k_j^0 V}. \quad (13.18)$$

This is the probability per unit time to scatter into a set of outgoing particles with precise momenta  $k_j^{\prime\mu}$ . To get something measurable we want a differential cross-section  $d\sigma$ . The first step is to integrate over outgoing three-momenta  $\mathbf{k}'_j$  over some small range. Due to the box all three-momenta are quantized:  $\mathbf{k}'_j = (2\pi/L)\mathbf{n}'_j$ , where  $V = L^3$  and  $\mathbf{n}'_j$  is a three-vector with integer entries (we are assuming periodic boundary conditions). In the limit of large  $L$  we have

$$\sum_{\mathbf{n}'_j} \longrightarrow \frac{V}{(2\pi)^3} \int d^3k'_j, \quad (13.19)$$

so we should multiply  $\dot{P}$  by one of these factors for each outgoing particle. This gives

$$\dot{P} \longrightarrow \frac{(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}})}{4E_1 E_2 V} |\mathcal{T}|^2 \prod_{j=1}^{n'} \frac{d^3k'_j}{(2\pi)^3 2k_j^0}. \quad (13.20)$$

To finally convert  $\dot{P}$  into  $d\sigma$ , we have to divide by the incident flux. We will do this in the FT frame where particle 2 is at rest. The incident flux is then the sum of the speeds of the

incident particles striking particle 2, per unit volume. We just have one incident particle, of speed  $|\mathbf{k}_1|/E_1$ , so the incident flux is  $|\mathbf{k}_1|/(E_1V)$ . We divide (13.20) by this and set  $E_2 = m_2$  to get

$$d\sigma = \frac{(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}})}{4|\mathbf{k}_1|_{FT} m_2} |\mathcal{T}|^2 \prod_{j=1}^{n'} \frac{d^3 k'_j}{(2\pi)^3 2k'_j{}^0}, \quad \text{FT frame} \quad (13.21)$$

We want to upgrade this to something valid in an arbitrary frame. To do this notice that the expression for  $|\mathbf{k}_1|_{FT} m_2$  in (13.10) is actually written generally in terms of the Lorentz invariant  $s$ . So to upgrade this to a Lorentz-invariant expression we simply use that expression. As a convention, we will actually use (13.11) to swap out for  $|\mathbf{k}_1|_{CM} \sqrt{s}$ , but again this expression has a Lorentz-invariant form given by the right-hand-side of (13.2). So in the end we have

$$d\sigma = \frac{1}{4|\mathbf{k}_1|_{CM} \sqrt{s}} |\mathcal{T}|^2 d\text{LIPS}_{n'}(k_1 + k_2), \quad (13.22)$$

where we  $|\mathbf{k}_1|_{CM}$  is given in terms of  $s$  by (13.2), and we have defined the  $n'$ -body Lorentz-invariant phase space measure

$$d\text{LIPS}_{n'}(k) := (2\pi)^4 \delta^4(k - \sum_{j=1}^{n'} k'_j) \prod_{j=1}^{n'} \frac{d^3 k'_j}{(2\pi)^3 2k'_j{}^0}. \quad (13.23)$$

Well, this  $d\text{LIPS}$  name is awful, but besides that, we have our final result for the differential cross section for the scattering  $2 \rightarrow n'$ .

We can specify now to  $n' = 2$ . We need to evaluate

$$d\text{LIPS}_2(k_1 + k_2) = (2\pi)^4 \delta^4(k_1 + k_2 - k'_1 - k'_2) \frac{d^3 k'_1}{(2\pi)^3 2k'_1{}^0} \frac{d^3 k'_2}{(2\pi)^3 2k'_2{}^0}. \quad (13.24)$$

We can do it in any reference frame since it is Lorentz-invariant. We'll pick the CM frame where  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$  and  $k_1^0 + k_2^0 = E_1 + E_2 = \sqrt{s}$ . Then we can write

$$d\text{LIPS}_2(k_1 + k_2) = \frac{d^3 k'_1 d^3 k'_2}{4(2\pi)^2 E'_1 E'_2} \delta(E'_1 + E'_2 - \sqrt{s}) \delta^3(\mathbf{k}'_1 + \mathbf{k}'_2) \quad (13.25)$$

We integrate over  $d^3 k'_2$  to get

$$d\text{LIPS}_2(k_1 + k_2) = \frac{d^3 k'_1}{4(2\pi)^2 E'_1 E'_2} \delta(E'_1 + E'_2 - \sqrt{s}) \quad (13.26)$$

where now

$$E'_1 = \sqrt{\mathbf{k}'_1{}^2 + m_1'^2}, \quad E'_2 = \sqrt{\mathbf{k}'_1{}^2 + m_2'^2} \quad (13.27)$$

due to the delta function. We now write

$$d^3k'_1 = |\mathbf{k}'_1|^2 d|\mathbf{k}'_1| \sin \theta d\theta d\phi =: |\mathbf{k}'_1|^2 d|\mathbf{k}'_1| d\Omega_{CM} \quad (13.28)$$

where  $d\Omega_{CM}$  is the differential solid angle, with  $\theta$  the angle between  $\mathbf{k}_1$  and  $\mathbf{k}'_1$  in the CM frame. We can now do the  $d|\mathbf{k}'_1|$  integral as well, using the delta function identity  $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ , where the sum is over the zeroes of  $f(x)$ . Our delta function argument has just one zero, it is when  $|\mathbf{k}_1|$  is given by (13.2). Using (13.27),  $f'(x_i)$  therefore evaluates to

$$\frac{\partial}{\partial |\mathbf{k}'_1|} (E'_1 + E'_2 - \sqrt{s}) = \frac{|\mathbf{k}'_1|}{E'_1} + \frac{|\mathbf{k}'_1|}{E'_2} = |\mathbf{k}'_1| \left( \frac{E'_1 + E'_2}{E'_1 E'_2} \right) = \frac{|\mathbf{k}'_1| \sqrt{s}}{E'_1 E'_2}. \quad (13.29)$$

We finally have

$$d\text{LIPS}_2(k_1 + k_2) = \frac{|\mathbf{k}'_1|}{16\pi^2 \sqrt{s}} d\Omega_{CM}. \quad (13.30)$$

Our differential cross-section is therefore

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} \frac{|\mathbf{k}'_1|}{|\mathbf{k}_1|} |\mathcal{T}|^2, \quad (13.31)$$

where  $|\mathbf{k}_1|$  and  $|\mathbf{k}'_1|$  are the functions of  $s$  given by (13.2) - (13.3), and  $d\Omega_{CM} = \sin \theta d\theta d\phi$  where  $\theta$  is the angle between  $\mathbf{k}_1$  and  $\mathbf{k}'_1$  in the CM frame.

We can upgrade this final expression to a Lorentz-invariant form as follows. We take the differential of the Mandelstam  $t$  at fixed  $s$  using (13.4) to get

$$dt = 2|\mathbf{k}_1||\mathbf{k}'_1| d \cos \theta = 2|\mathbf{k}_1||\mathbf{k}'_1| \frac{d\Omega_{CM}}{2\pi}, \quad (13.32)$$

which lets us rewrite (13.31) as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\mathbf{k}_1|^2} |\mathcal{T}|^2, \quad (13.33)$$

where as usual  $|\mathbf{k}_1|$  is given as a function of  $s$  by (13.2). We can use this to go into any frame we wish by taking the differential of (13.4) in that frame, although generally this will

be more complicated than the CM frame.

We can define a total cross section  $\sigma$  by integrating the differential one over all possible momenta. Let's do this for  $n'$  outgoing particles. We will have to divide by an appropriate symmetry factor in the case of identical outgoing particles:

$$S = \prod_i n'_i!, \quad (13.34)$$

where there are  $n'_i$  identical outgoing particles of type  $i$ . This symmetry factor is needed because the  $d\text{LIPS}$  outgoing momentum integrals are all treated distinctly, which is incorrect if some of the particles are identical. Thus we have

$$\sigma = \frac{1}{S} \int d\sigma. \quad (13.35)$$

For two outgoing particles we have

$$\sigma = \frac{1}{S} \int d\Omega_{CM} \frac{d\sigma}{d\Omega_{CM}} = \frac{2\pi}{S} \int_{-1}^{+1} d \cos \theta \frac{d\sigma}{d\Omega_{CM}}. \quad (13.36)$$

We can also write this as

$$\sigma = \frac{1}{S} \int_{t_{\min}}^{t_{\max}} dt \frac{d\sigma}{dt}. \quad (13.37)$$

To do this integral, we should trade out  $u$  for  $s$  and  $t$  using (13.8) and then integrate over  $t$  at fixed  $s$ .  $t_{\min}$  is fixed by  $\cos \theta = -1$  and  $t_{\max}$  by  $\cos \theta = +1$ .

Let's do this for  $\phi^3$  theory. We need to express  $\mathcal{T}$  from (13.9) in terms of  $t$  and  $\sigma$  to use (13.37). We use that  $t_{\min} = -(s - 4m^2)$  and  $t_{\max} = 0$ . We use the linear constraint (13.8) to write  $u = 4m^2 - s - t$ , which gives

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\mathbf{k}_1|^2} |\mathcal{T}|^2 = \frac{\lambda^4}{16\pi s (s - 4m^2)} \left( \frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{s + t - 3m^2} \right)^2 + O(\lambda^6). \quad (13.38)$$

To get the total cross section we use that the outgoing particles are identical so  $S = 2$ . Integrating then gives

$$\sigma = \frac{\lambda^4}{32\pi s (s - 4m^2)} \left( \frac{2}{m^2} + \frac{s - 4m^2}{(s - m^2)^2} - \frac{2}{s - 3m^2} + \frac{4m^2}{(s - m^2)(s - 2m^2)} \log \left( \frac{s - 3m^2}{m^2} \right) \right) \quad (13.39)$$

$$+ O(\lambda^6) \quad (13.40)$$

In the nonrelativistic limit  $s - 4m^2 \ll m^2$  this becomes

$$\sigma = \frac{25\lambda^4}{1152\pi m^6} \left( 1 - \frac{79}{60} \left( \frac{s - 4m^2}{m^2} \right) + \dots \right) + O(\lambda^6) \quad \text{nonrelativistic,} \quad (13.41)$$

while in the extreme relativistic limit  $s \gg m^2$  this becomes

$$\sigma = \frac{\lambda^4}{16\pi m^2 s^2} \left( 1 + \frac{7}{2} \frac{m^2}{s} + \dots \right) + O(\lambda^6) \quad \text{extreme relativistic.} \quad (13.42)$$

*Example 1:* Let's compute the cross-section in the CM frame for  $\phi^3$  theory, which requires using (13.36). To use it, we need to express  $t$  and  $u$  in terms of  $s$  and  $\theta$  and then integrate over  $q$  at fixed  $s$ .

$\mathcal{T}$  is given by (13.9). Since all the masses are equal, in the CM frame we have  $E = \sqrt{s}/2$  for the energy of any of the four particles, and (13.2) - (13.3) simplify to  $|\mathbf{k}'_1| = |\mathbf{k}_1| = \frac{1}{2}\sqrt{s - 4m^2}$ . Then (13.4) becomes

$$t = -\frac{1}{2}(s - 4m^2)(1 - \cos \theta). \quad (13.43)$$

The constraint (13.8) gives

$$u = -\frac{1}{2}(s - 4m^2)(1 + \cos \theta). \quad (13.44)$$

So  $|\mathcal{T}|^2$  is a bit of a mess. We can consider the nonrelativistic limit  $|\mathbf{k}_1| \ll m$ , which from the above gives  $s - 4m^2 \ll m^2$ . Thus

$$\mathcal{T} = \frac{5\lambda^2}{3m^2} \left[ 1 - \frac{8}{15} \left( \frac{s - 4m^2}{m^2} \right) + \frac{5}{18} \left( 1 + \frac{27}{25} \cos^2 \theta \right) \left( \frac{s - 4m^2}{m^2} \right)^2 + \dots \right] + O(\lambda^4). \quad (13.45)$$

At leading order in the nonrelativistic expansion we see that the differential cross-section is isotropic.

We can also consider the extreme relativistic limit  $|\mathbf{k}_1| \gg m$ , i.e.  $s \gg m^2$ , which gives

$$\mathcal{T} = \frac{\lambda^2}{s \sin^2 \theta} \left[ 3 + \cos^2 \theta - \left( \frac{(3 + \cos^2 \theta)^2}{\sin^2 \theta} - 16 \right) \frac{m^2}{s} + \dots \right] + O(g^4). \quad (13.46)$$

Notice that the differential cross section in this limit is sharply peaked in the forward ( $\theta = 0$ )

and backward ( $\theta = \pi$ ) directions.

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We will actually not treat decay rates (e.g. 1-to- $n'$  processes for  $n' > 1$ ), since there is a subtlety with the LSZ formula in this case: the formula uses exact eigenstates of the full Hamiltonian for the incoming states, but such things are stable and do not decay! See Chapter 25 of [1] for some more discussion about this, although it is not totally satisfactory.

## 14 Dimensional analysis

Since we will begin doing things in general dimension, it will help to have a handle on the units of various quantities and their dependence on dimension. We are working with  $\hbar = c = 1$ . That means time  $T$  and length  $L$  are the same units (since they are normally converted by  $L = vT$  for some velocity  $v$  which is now dimensionless), and length and inverse mass  $M^{-1}$  are the same units (to see how they are converted, consider the formulas  $E = hc/\lambda$  and  $E = Mc^2$  where  $\lambda$  is a length). So any quantity  $A$  can be thought of as having units of mass to some power. We will call this power  $[A]$ . We have

$$[m] = +1, \quad [x^\mu] = -1, \quad [\partial^\mu] = +1, \quad [d^d x] = -d. \quad (14.1)$$

We consider a general interacting scalar field theory in  $d$  dimensions

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \sum_{n=3}^N \frac{\lambda_n}{n!}\phi^n. \quad (14.2)$$

The action is  $S = \int d^d x \mathcal{L}$  and the path integral is

$$Z(J) = \int D\phi \exp \left[ i \int d^d x (\mathcal{L} + J\phi) \right]. \quad (14.3)$$

The action must be dimensionless, since it sits in an exponent, so  $[S] = 0$  (recall that it's really  $e^{iS/\hbar}$  but we have set  $\hbar = 1$ ). We therefore have  $[\mathcal{L}] = d$ , so every term in the Lagrangian must have this mass dimension. Investigating  $\partial^\mu\partial_\mu\phi$ , we see that

$$[\phi] = \frac{1}{2}(d-2). \quad (14.4)$$

Then investigating  $\lambda_n\phi^n$ , we see that

$$[\lambda_n] = d - \frac{n}{2}(d-2). \quad (14.5)$$

For our  $\phi^3$  theory, we have

$$[\lambda_3] = \frac{1}{2}(6-d). \quad (14.6)$$

Notice it is dimensionless for  $d = 6$ . Theories with dimensionless couplings have much more structure at all energy scales, so we will fix to this case later.

Coupling with mass dimension  $[\lambda] < d$ ,  $\lambda = d$ , and  $\lambda > d$  are called relevant, marginal,

and irrelevant, respectively. You will soon learn that relevant couplings are important at low energies (and unimportant at high energies), marginal couplings are important at all energies, and irrelevant couplings are unimportant at low energies. Since marginal couplings have this rich structure, we will at various points stick  $d = 6$  in our calculations in  $\phi^3$  theory.

## 15 The Kallen-Lehmann form of the exact propagator

In the next few sections we will be studying loop corrections to vertices and propagators. When we get into the guts of the calculations, we will often fix the spacetime dimension  $d = 6$  for our  $\phi^3$  theory, since for reasons you will learn about next quarter this is the simplest situation to analyze the effects we are interested in. While setting up formalism, however, we will keep  $d$  general.

The first thing we'll do is see how much we can learn about  $G^{\text{exact}}(x - y)$  from general principles. Recall that

$$G^{\text{exact}}(x - y) := i\langle 0|T\phi(x)\phi(y)|0\rangle. \quad (15.1)$$

According to our LSZ analysis, we want normalization

$$\langle 0|\phi(x)|0\rangle = 0, \quad \langle k|\phi(x)|0\rangle = e^{-ikx}. \quad (15.2)$$

The one-particle momentum eigenstates  $|k\rangle$  have normalization

$$\langle k|k'\rangle = (2\pi)^{d-1}2\omega\delta^{d-1}(\mathbf{k} - \mathbf{k}'), \quad (15.3)$$

with  $\omega = \sqrt{\mathbf{k}^2 + m^2}$  as usual. (We will work in our on-shell renormalization scheme where  $Z = 1$  and  $m = m_{\text{ph}}$ .) The identity operator in the one-particle subspace has resolution

$$I_1 = \int \widetilde{dk} |k\rangle\langle k|, \quad (15.4)$$

where

$$\widetilde{dk} := \frac{d^{d-1}k}{(2\pi)^{d-1}2\omega} \quad (15.5)$$

is the Lorentz invariant phase-space differential. We can also Fourier transform

$$G^{\text{exact}}(x - y) = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \tilde{G}^{\text{exact}}(k^2). \quad (15.6)$$

Let's warm up with the free theory, for which

$$\tilde{G}^{\text{exact}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon}. \quad (15.7)$$

This object has a single pole at  $k^2 = -m^2$ , with residue one.  $m$  is the physical mass of the

particle, i.e. it is what enters into the four-momentum of the particle.

Let's now analyze a general interacting scalar field theory. We assume there exists a vacuum state with vanishing energy and momentum. Let's analyze the states with vanishing 3-momentum. The lowest energy state after the vacuum is a single-particle state. This has energy  $E = m$ . So there is a gap above the vacuum state. We will assume that there are no bound states in the theory, in which case the next set of states is a continuum which begins at  $E = 2m$ . It is a continuum because even with vanishing total 3-momentum the multiple particles can have relative momenta which can lead to any energy we wish. And of course, we can move off the axis of vanishing total 3-momentum.

To analyze the exact propagator, let's take  $x^0 > y^0$  so we can ignore the  $T$  symbol. We will insert a complete set of energy eigenstates (of the type discussed immediately above) between the two fields. Our notation will be  $|0\rangle$  for the vacuum,  $|k\rangle$  for single-particle states specified by a 3-momentum  $\mathbf{k}$  and energy  $\omega = \sqrt{\mathbf{k}^2 + m^2}$ , and  $|k, n\rangle$  for multiparticle states (which form a continuum) with total 3-momentum  $\mathbf{k}$  and other parameters, like relative momenta, that we will collectively denote by  $n$ . The energy of one of these states is  $\omega = \sqrt{\mathbf{k}^2 + M^2}$  where  $M \geq 2m$  is one of the parameters in the set  $n$ . So we have

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle + \int \widetilde{dk} \langle 0|\phi(x)|k\rangle\langle k|\phi(y)|0\rangle \quad (15.8)$$

$$+ \sum_n \int \widetilde{dk} \langle 0|\phi(x)|k, n\rangle\langle k, n|\phi(y)|0\rangle \quad (15.9)$$

The sum over  $n$  is schematic and represents multiple things like integrals over relative momenta. Using (15.2) and

$$\langle k, n|\phi(x)|0\rangle = \langle k, n|e^{-iP^\mu x_\mu}\phi(0)e^{iP^\mu x_\mu}|0\rangle = e^{-ikx}\langle k, n|\phi(0)|0\rangle \quad (15.10)$$

gives

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \widetilde{dk} e^{ik(x-y)} + \sum_n \int \widetilde{dk} e^{ik(x-y)} |\langle 0|\phi(0)|k, n\rangle|^2. \quad (15.11)$$

Let's cook up some notation. We define the *spectral density*

$$\rho(s) := \sum_n |\langle k, n|\phi(0)|0\rangle|^2 \delta(s - M^2) \quad (15.12)$$

Since  $M$  is one of the parameters in  $n$ , and the multiparticle continuum begins at  $2m$ , i.e.

$s = 4m^2$ , we have  $\rho(s) \geq 0$  for  $s \geq 4m^2$  and  $\rho(s) = 0$  for  $s < 4m^2$ . So we can write

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \widetilde{d}k e^{ik(x-y)} + \int_{4m^2}^{\infty} ds \rho(s) \int \widetilde{d}k e^{ik(x-y)}. \quad (15.13)$$

Since the first term came from the single-particle states it has  $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ , while the second term has  $k^0 = \sqrt{\mathbf{k}^2 + M^2} = \sqrt{\mathbf{k}^2 + s}$  due to the delta function. Considering the other time-ordering gives simply flips  $x \leftrightarrow y$ , which we can combine to write

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle. \quad (15.14)$$

Now comes the key step. We use our old expression for the Feynman propagator

$$\int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} = i\theta(x^0 - y^0) \int \widetilde{d}k e^{ik(x-y)} + i\theta(y^0 - x^0) \int \widetilde{d}k e^{-ik(x-y)}, \quad (15.15)$$

obtained by closing the contour and integrating, to write

$$i\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \left[ \frac{1}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} ds \rho(s) \frac{1}{k^2 + s - i\epsilon} \right]. \quad (15.16)$$

We can now identify the Fourier transform as

$$\tilde{G}^{\text{exact}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} ds \rho(s) \frac{1}{k^2 + s - i\epsilon}. \quad (15.17)$$

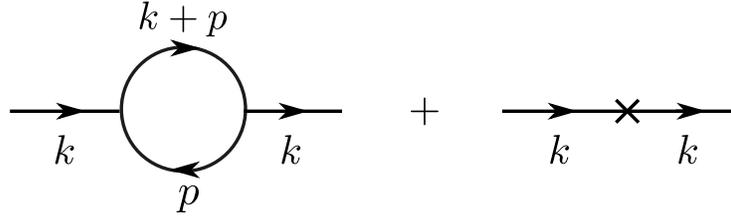
This is it! This is known as the Kallen-Lehmann spectral representation of the exact momentum-space propagator. A key result is that  $\tilde{G}^{\text{exact}}(k^2)$  has an isolated pole at  $k^2 = -m^2$  with residue one, just like the propagator in free-field theory. In a more general renormalization scheme this first term would look like  $Z/(k^2 + m_{\text{ph}}^2 - i\epsilon)$ . Notice that this representation of the scalar propagator is *theory-independent* – all of the content of the particular theory one is considering is buried in the spectral density  $\rho(s)$ , and in the mass  $m$  of a single-particle state.

## 16 Loop corrections to the propagator

Another way we could try to compute the exact propagator is by summing up all the loop corrections to the free propagator. This should agree with our general formula above, so let's check that it does. We should pick a particular theory to calculate in. We will, of course, pick  $\phi^3$  theory. From now on we will call the field  $\phi$  instead of  $\phi_r$  to reduce clutter, but keep in mind that it satisfies the LSZ conditions (15.2). So we want to calculate

$$\frac{1}{i}G^{\text{exact}}(x_1 - x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \delta_1\delta_2iW(J)\Big|_{J=0}. \quad (16.1)$$

Recall  $W(J)$  is the sum of connected diagrams with  $W(0) = 0$ , i.e. there are no vacuum diagrams. The  $O(\lambda^2)$  corrections to the propagator are given by



Using the momentum-space Feynman rules we wrote down previously gives (recall we are back to using the Feynman rules for time-ordered correlators, not scattering amplitudes)

$$\frac{1}{i}\tilde{G}^{\text{exact}}(k^2) = \frac{1}{i}\tilde{G}(k^2) + \frac{1}{i}\tilde{G}(k^2) [i\Pi(k^2)] \frac{1}{i}\tilde{G}(k^2) + O(\lambda^4), \quad (16.2)$$

where

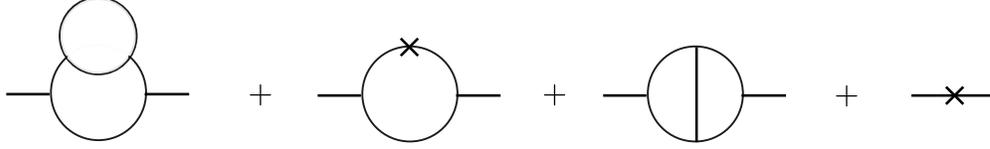
$$\tilde{G}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} \quad (16.3)$$

is the free-field propagator and we have defined

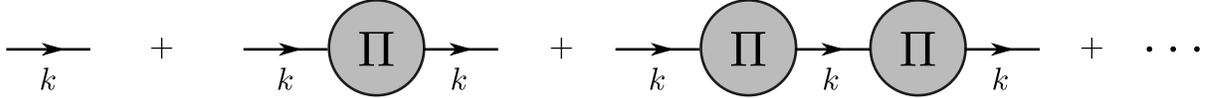
$$i\Pi(k^2) = \frac{1}{2}(i\lambda)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^d p}{(2\pi)^d} \tilde{G}((k+p)^2)\tilde{G}(p^2) - i(\delta_Z k^2 + \delta_m) + O(\lambda^4), \quad (16.4)$$

known as the *self-energy*. The  $1/2$  is due to a symmetry of the bubble: exchanging the two propagators in it and simultaneously two of the three  $\delta/\delta J$ 's in each of the clusters leaves the diagram unchanged. Recall that  $\delta_Z, \delta_m \sim O(\lambda^2)$ . Notice that we don't have a counterterm three-point vertex with a similar bubble structure since  $\delta_\lambda \sim O(\lambda^2)$ , so the contribution of that diagram would be  $O(\lambda^4)$ .

We can calculate  $\tilde{G}^{\text{exact}}(k^2)$  to all orders in  $\lambda$  if we consider higher-order corrections to  $\Pi(k^2)$ , e.g. the  $O(\lambda^4)$  corrections below



and if we consider arbitrarily many insertions of  $\Pi$ , as below



In equations, this diagram says

$$\frac{1}{i}\tilde{G}^{\text{exact}}(k^2) = \frac{1}{i}\tilde{G}(k^2) + \frac{1}{i}\tilde{G}(k^2) [i\Pi(k^2)] \frac{1}{i}\tilde{G}(k^2) \quad (16.5)$$

$$+ \frac{1}{i}\tilde{G}(k^2) [i\Pi(k^2)] \frac{1}{i}\tilde{G}(k^2) [i\Pi(k^2)] \frac{1}{i}\tilde{G}(k^2) \quad (16.6)$$

$$+ \dots \quad (16.7)$$

If we take  $i\Pi(k^2)$  to be the sum of all possible “one-particle irreducible” (1PI for short) diagrams, then this sum will indeed include all contributions to  $\tilde{G}^{\text{exact}}(k^2)$ . 1PI diagrams are ones that cannot be disconnected into two pieces by cutting a single line. All 1PI diagrams that contribute to  $i\Pi(k^2)$  at  $O(\lambda^4)$  are exhibited in the set of diagrams above. The contributions to  $i\Pi(k^2)$  are defined without including the external propagators (since they will be included in the sum for the propagator). Summing the geometric series above gives

$$\tilde{G}^{\text{exact}}(k^2) = \frac{\tilde{G}(k^2)}{1 - i\Pi(k^2)\frac{1}{i}\tilde{G}(k^2)} = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)}. \quad (16.8)$$

This is yet another general representation of the scalar propagator, where now the content of the particular theory being considered is buried in  $\Pi(k^2)$  and the mass of single-particle states  $m$ . But this should definitely agree with our Kallen-Lehmann representation (15.17), so we should be able to write it in that form. This means that in our renormalization scheme the expression above should have a pole at  $k^2 = -m^2$  with residue one. This is true if and

only if

$$\Pi(-m^2) = 0, \quad \Pi'(-m^2) = 0, \quad (16.9)$$

where the prime denotes a derivative with respect to  $k^2$ . Practically speaking, (16.9) is how we will implement our renormalization conditions. We will therefore use it to solve for  $\delta_Z$  and  $\delta_m$ .

Having shown the consistency between summing all loop diagrams and the general approach from the previous section, let's now try to calculate an actual correction to the propagator. To calculate the  $O(\lambda^2)$  corrections, we need to evaluate the integrals in (16.4). Notice that for  $d \geq 4$ , the momentum integral over  $p$  diverges for large  $p$  (logarithmically for  $d = 4$ ). We can introduce an ultraviolet cutoff, as we did for the tadpole diagrams, but let's just proceed and see what we can say about the formal structure of this integral.

To evaluate the integral we need several tricks. The first is Feynman's formula to combine denominators,

$$\frac{1}{A_1 \cdots A_n} = \int dF_n (x_1 A_1 + \cdots + x_n A_n)^{-n}, \quad (16.10)$$

where the integration measure is over the "Feynman parameters"  $x_i$ :

$$\int dF_n = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(x_1 + \cdots + x_n - 1), \quad \int dF_n = 1. \quad (16.11)$$

We use this to write

$$\tilde{G}((k+p)^2) \tilde{G}(p^2) = \frac{1}{(p^2 + m^2)((p+k)^2 + m^2)} \quad (16.12)$$

$$= \int_0^1 dx [x((p+k)^2 + m^2) + (1-x)(p^2 + m^2)]^{-2} \quad (16.13)$$

$$= \int_0^1 dx [p^2 + 2xp \cdot k + xk^2 + m^2]^{-2} \quad (16.14)$$

$$= \int_0^1 dx [(p+xk)^2 + x(1-x)k^2 + m^2]^{-2} \quad (16.15)$$

$$= \int_0^1 dx [q^2 + D]^{-2}, \quad (16.16)$$

where  $q := p + xk$  and  $D := x(1-x)k^2 + m^2$ . We have suppressed the  $i\epsilon$  factors although they are important here. In particular, we want to change integration variables from  $p$  to  $q$  (which has trivial Jacobian), and to do the integral over  $q^0$  we want to rotate the contour

$q^0 \in (-\infty, \infty)$  to the imaginary axis  $q^0 \in (-i\infty, i\infty)$ . We can do this if the integrand decays sufficiently quickly for  $|q^0| \rightarrow \infty$ , and if we do not pass through any poles. The poles are at  $q^0 = -\omega + i\epsilon$  and  $q^0 = \omega - i\epsilon$ , so we simply need to rotate the contour counterclockwise! We will implement this by the analytic continuation  $q^0 = iq_d$  (with  $q_j$  unchanged). Notice by this definition the Lorentzian 4-vector  $q^2 = -(q^0)^2 + q_i^2$  equals the Euclidean 4-vector  $q_E^2 = q_i^2 + q_d^2$  where  $i = 1, \dots, d-1$ . So all that changes is the measure  $d^d q \rightarrow id^d q_E$ .

Defining

$$I(k^2) := \int_0^1 dx \int \frac{d^d q_E}{(2\pi)^d} \frac{1}{(q_E^2 + D)^2} \quad (16.17)$$

lets us write

$$\Pi(k^2) = \frac{1}{2}\lambda^2 I(k^2) - \delta_Z k^2 - \delta_m + O(\lambda^4). \quad (16.18)$$

We can now evaluate  $I$ . We actually want the result for  $d = 6$ , but the integral looks divergent for  $d \geq 4$ . We can deal with this in many ways. One is dimensional regularization: evaluate the integral for arbitrary  $d < 4$  and then analytically continue in dimension. Alternatively, we could introduce an ultraviolet cutoff as we saw before in the evaluation of  $G(0)$ . We would do this via the replacement

$$\tilde{G}(\ell^2) \rightarrow \frac{1}{\ell^2 + m^2 - i\epsilon} \frac{\Lambda^2}{\ell^2 + \Lambda^2 - i\epsilon} \quad (16.19)$$

in the expression  $\tilde{G}((k+p)^2)\tilde{G}(p^2)$ . This is known as Pauli-Villars regularization, and makes the integral convergent for  $d < 8$ .

Why do these work, and how do we know they won't give different answers? To see this, consider expanding  $\Pi(k^2)$  around  $-m^2$ :

$$\Pi(k^2) = \left[ \frac{1}{2}\lambda^2 I(-m^2) + \delta_Z m^2 - \delta_m \right] \quad (16.20)$$

$$+ \left[ \frac{1}{2}\lambda^2 I'(-m^2) + \delta_Z \right] (k^2 + m^2) \quad (16.21)$$

$$+ \frac{1}{2!} \left[ \frac{1}{2}\lambda^2 I''(-m^2) \right] (k^2 + m^2)^2 + \dots + O(\lambda^4) \quad (16.22)$$

From the explicit integral expression (16.17) we see that  $I(-m^2)$  diverges for  $d \geq 4$ ,  $I'(-m^2)$  diverges for  $d \geq 6$ , and in general  $I^{(n)}(-m^2)$  diverges for  $d \geq 4 + 2n$ . Since  $\delta_Z$  and  $\delta_m$  begin at  $O(\lambda^2)$ , we can use them to cancel the first two divergences,  $\frac{1}{2}\lambda^2 I(-m^2)$  and  $\frac{1}{2}\lambda^2 I'(-m^2)$ .

This will require the counterterms be infinite, but we've seen this before. The counterterms are not directly measurable, and their values can differ depending on how we regularize these infinities (i.e. Pauli-Villars vs dimensional regularization). Of course, to have a completely finite expression the rest of the  $I^{(n)}(-m^2)$  need to be finite as well. This is only true for  $d < 8$ , where  $I''(-m^2)$  and higher derivatives give manifestly convergent integrals. The fact that our counterterms yield a finite expression for  $d < 8$  is precisely why Pauli-Villars worked to give a finite expression for  $d < 8$ . In this range, any regularization which preserves Lorentz invariance will give the same answer.

Not convinced that these two methods (and others) will agree? Here is another argument. Let's compute  $\Pi(k^2) = \frac{1}{2}\lambda^2 I(k^2) - \delta_Z k^2 - \delta_m + O(\lambda^2)$  by first taking two derivatives of  $\Pi(k^2)$  with respect to  $k^2$ :

$$\Pi''(k^2) = \frac{1}{2}\lambda^2 I''(k^2) + O(\lambda^4), \quad I''(k^2) = \int_0^1 dx 6x^2(1-x)^2 \frac{d^d q_E}{(2\pi)^d} \frac{1}{(q_E + D)^4}. \quad (16.23)$$

This is now convergent for  $d < 8$ , and we can integrate it up using the boundary conditions  $\Pi(-m^2) = \Pi'(-m^2) = 0$  to get  $\Pi(k^2)$ , without even computing the counterterms!

Enough philosophy, let's calculate  $I(k^2)$ . We work in spherical coordinates, and the angular part of the integrand gives the area of the  $d$ -dimensional unit sphere  $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ . The radial part of the integral can be evaluated in terms of gamma functions. We will use the following general expression

$$\int \frac{d^d q_E}{(2\pi)^d} \frac{(q_E^2)^a}{(q_E^2 + D)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)} D^{-(b-a-d/2)}, \quad (16.24)$$

where our case is  $a = 0$  and  $b = 2$ .

Now we want to fix to  $d = 6$ , since in that case  $\lambda$  is dimensionless. We will actually put in  $d = 6 - \varepsilon$  and expand in  $\varepsilon$ . In this case  $\lambda$  has mass dimension  $\varepsilon/2$ . At this point it is helpful to introduce an (arbitrary!!) mass parameter  $\tilde{\mu}$ , with  $[\tilde{\mu}] = 1$ , and measure dimensionful quantities in units of this mass parameter. For example we define the dimensionless coupling

$$\bar{\lambda} = \frac{\lambda}{\tilde{\mu}^{\varepsilon/2}}. \quad (16.25)$$

$\bar{\lambda}$  is just the original dimensionful coupling measured in units of  $\tilde{\mu}$ . So  $[\bar{\lambda}] = 0$ . This is just bookkeeping: since  $\tilde{\mu}$  was arbitrary, our scattering amplitudes will not depend on it.<sup>24</sup>

<sup>24</sup> [1] glosses over this point, but it is actually quite subtle. Let's say we are in the situation with  $c \neq 1$  and we take a velocity  $v$  with units of  $m/s$  and define a dimensionless velocity  $\bar{v} = v/c$ . Now we ask: "how long would it take to travel a distance  $L$  when going at velocity  $v$ ?" Phrased in terms of  $\bar{v}$ , the answer is  $t = L/(\bar{v}c)$ , so the answer clearly depends on  $c$ . If we defined a different  $\bar{v} = v/v_s$ , where  $v_s$  is say the speed

Using (16.24) with  $a = 0$ ,  $b = 2$  to evaluate (16.17) and set  $d = 6 - \varepsilon$  gives

$$I(k^2) = \frac{\Gamma(-1 + \varepsilon/2)}{(4\pi)^3} \int_0^1 dx D \left( \frac{4\pi}{D} \right)^{\varepsilon/2}. \quad (16.26)$$

Defining  $\alpha = \bar{\lambda}^2/(4\pi)^3$  lets us write the self-energy as

$$\Pi(k^2) = \frac{\alpha}{2} \Gamma(-1 + \varepsilon/2) \int_0^1 dx D \left( \frac{4\pi\tilde{\mu}^2}{D} \right)^{\varepsilon/2} - \delta_Z k^2 - \delta_m + O(\alpha^2). \quad (16.27)$$

Taking the  $\varepsilon \rightarrow 0$  limit with the help of  $X^{\varepsilon/2} = 1 + \frac{\varepsilon}{2} \ln X + O(\varepsilon^2)$  and  $\Gamma(-n + \varepsilon/2) = \frac{(-1)^n}{n!} \left( 2/\varepsilon - \gamma + \sum_{j=1}^n j^{-1} + O(\varepsilon) \right)$  for Euler-Mascheroni constant  $\gamma = 0.5772\dots$  gives

$$\Pi(k^2) = -\frac{\alpha}{2} \left[ \left( \frac{2}{\varepsilon} + 1 \right) \left( \frac{k^2}{6} + m^2 \right) + \int_0^1 dx D \ln \left( \frac{4\pi\tilde{\mu}^2}{e^\gamma D} \right) \right] - \delta_Z k^2 - \delta_m + O(\alpha^2), \quad (16.28)$$

where we did the trivial integral  $\int_0^1 dx D = k^2/6 + m^2$ . To clean things up a bit more we define  $\mu = \sqrt{4\pi} e^{-\gamma/2} \tilde{\mu}$  to write

$$\Pi(k^2) = \frac{\alpha}{2} \int_0^1 dx D \ln(D/m^2) - \left\{ \frac{\alpha}{6} \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} \right] + \delta_Z \right\} k^2 \quad (16.29)$$

$$- \left\{ \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} \right] + \delta_m/m^2 \right\} m^2 + O(\alpha^2). \quad (16.30)$$

Now we can pick  $\delta_Z$  and  $\delta_m$  to cancel the divergences and  $\mu$  dependence:

$$\delta_Z = -\frac{\alpha}{6} \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} + \kappa_Z \right] + O(\alpha^2), \quad (16.31)$$

$$\delta_m = -m^2 \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} + \kappa_m \right] + O(\alpha^2), \quad (16.32)$$

---

of sound, then the answer is  $t = L/(\bar{v}v_s)$ . So even though  $c$  or  $v_s$  or whatever was arbitrary, the answer clearly depends on it! But all  $c$  or  $v_s$  is doing is providing the right units for time. In our QFT analysis, recalling our  $c = \hbar = 1$  units, *everything* just has dimensions of mass to some power. This means we can take any observable quantity  $\sigma$  with mass dimension  $[\sigma] = k$  and write  $\sigma = m^k f(\dots)$  WLOG, where  $f(\dots)$  is some function of dimensionless parameters and  $m$  is some mass scale, we could choose it to be the mass of the particle. A dimensionless parameter is something like  $\lambda/m^{\varepsilon/2}$ . But notice if we write this in terms of dimensionless  $\bar{\lambda} = \lambda/\tilde{\mu}^{\varepsilon/2}$  and  $\bar{m} = m/\tilde{\mu}$ , then the  $\tilde{\mu}$  dependence drops out! This is what is meant by the fact that our physical answers cannot depend on this scale. (The overall  $m^k$  can be written as  $(\bar{m}\tilde{\mu})^k$  but this kind of overall dependence is not what we are interested in; we will leave such things simply as  $m^k$ .)

where  $\kappa_Z$  and  $\kappa_m$  are arbitrary constants. Thus we have

$$\Pi(k^2) = \frac{\alpha}{2} \int_0^1 dx D \ln(D/m^2) + \alpha \left( \frac{\kappa_Z}{6} k^2 + \kappa_m m^2 \right) + O(\alpha^2). \quad (16.33)$$

Finally, to fix  $\kappa_Z$  and  $\kappa_m$  we must impose the conditions  $\Pi(-m^2) = 0$  and  $\Pi'(-m^2) = 0$ . This can be done since the integral can be done in closed form, and gives

$$\kappa_Z = \frac{7}{3} - \frac{\sqrt{3}\pi}{2}, \quad \kappa_m = \frac{11}{12} - \frac{\sqrt{3}\pi}{6}. \quad (16.34)$$

Altogether this gives

$$\Pi(k^2) = \frac{\alpha}{12} \left[ (3 - \pi\sqrt{3})k^2 + (3 - 2\pi\sqrt{3})m^2 + 2k^2 r^3 \tanh^{-1}(1/r) \right] + O(\alpha^2), \quad (16.35)$$

where

$$r = \sqrt{1 + 4m^2/k^2}. \quad (16.36)$$

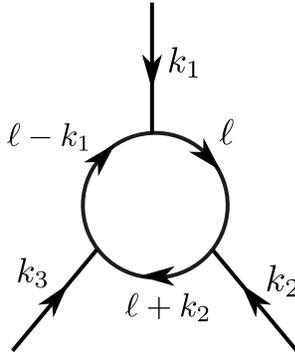
Phew! That is our final answer. As we discussed at the beginning of the section, our two expressions for the exact propagator (15.17) and (16.8) have to be consistent. We used this to extract some information about  $\Pi(k^2)$  evaluated at  $k^2 = -m^2$ . But now that we calculated  $\Pi(k^2)$  to  $O(\lambda^2)$ , we can compare the two expressions again and extract  $\rho(s)$  to  $O(\lambda^2)$ . The answer is

$$\pi\rho(s) = \frac{\text{Im } \Pi(-s)}{(-s + m^2 + \text{Re } \Pi(-s))^2 + (\text{Im } \Pi(-s))^2}. \quad (16.37)$$

For details see Chapter 15 of [1], but here we only want to point out that (15.17) and (16.8) are definitely consistent, and can be matched order by order in perturbation theory.

## 17 Loop corrections to the vertex

In the last section we studied perturbative corrections to the tree-level propagator. The tree-level propagator came from the term  $-\frac{1}{2}\partial_m\phi\partial^\mu\phi - m^2\phi^2/2$  in the Lagrangian. But there are also perturbative corrections to the tree-level vertex we can calculate. The tree-level vertex comes from the  $\lambda\phi^3/3!$  part of the Lagrangian and gives a factor of  $i\lambda$  in momentum space. A trivial correction to calculate is the counterterm three-point vertex, which comes from the term  $\delta_\lambda\phi^3/3!$  and is  $O(\lambda^2)$ . This is trivial to calculate because the Feynman rule tells us it simply contributes  $i\delta_\lambda$  in momentum space. It also doesn't help us fix  $\delta_\lambda$ , whereas in the last section we were able to fix  $\delta_Z$  and  $\delta_m$  by calculating perturbative corrections to the propagator. We should expect, generally, to fix the counterterms only when we run into diagrams with loop integrals that will diverge, because practically speaking the counterterms are there to deal with divergences. So let's go a bit further in our perturbation series. The next correction is  $O(\lambda^3)$  and is given by the figure below, which we will now calculate.



We can write the exact vertex as

$$iV_3^{\text{exact}}(k_1, k_2, k_3) = i\lambda + i\delta_\lambda = (i\lambda)^3 (-i)^3 \int \frac{d^d\ell}{(2\pi)^d} \tilde{G}((\ell - k_1)^2) \tilde{G}(\ell^2) \tilde{G}((\ell + k_2)^2) + O(\lambda^5) \quad (17.1)$$

Of course we have only explicitly written the vertex to  $O(\lambda^3)$ , with all the corrections that make this the exact vertex buried in  $O(\lambda^5)$ . If we again define  $\bar{\lambda} = \lambda/\tilde{\mu}^{\varepsilon/2}$ ,  $\alpha = \bar{\lambda}^2/(4\pi)^3$  and  $\mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2$  as in the previous section, we can write our answer as

$$\bar{\lambda}^{-1} V_3^{\text{exact}}(k_1, k_2, k_3) = 1 + \left\{ \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) \right] + \delta_\lambda \right\} - \frac{1}{2} \alpha \int dF_3 \ln(D/m^2) + O(\alpha^2) \quad (17.2)$$

where

$$\int dF_3 := 2 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1), \quad D := x_3 x_1 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2. \quad (17.3)$$

Evaluating the loop integral to get to this point is messy and uses some of the same tricks as we did for the propagator. It is an important skill to be able to evaluate these things efficiently, so I encourage you to work this out and consult Chapter 16 of [1] if you get stuck.

Now, as before, we can pick

$$\delta_\lambda = -\alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \kappa_\lambda \right] + O(\alpha^2), \quad (17.4)$$

for  $\kappa_\lambda$  purely a numerical constant, and we get a totally finite expression that is independent of  $\mu$ .

At this point in the propagator calculation, we had two free parameters  $\kappa_m$  and  $\kappa_Z$ , and we used our two conditions  $\Pi(-m^2) = \Pi'(-m^2) = 0$  to fix them. Those conditions were really renormalization conditions: we declared our Lagrangian mass parameter  $m$  to equal the physical or pole mass, and we declared the overlap to be normalized as  $\langle k | \phi(x) | 0 \rangle = e^{-ikx}$ . Here, to fix  $\kappa_\lambda$ , we need to define our coupling  $\lambda$  in some way. Remember: we want to compare to experiment to fix the parameters in our Lagrangian! Well, we can really define  $\lambda$  in terms of an experiment however we wish –  $\kappa_\lambda$  will just react to make sure the definition makes sense. For example, say we choose to define  $\lambda$  by the condition

$$V_3^{\text{exact}}(0, 0, 0) = \lambda. \quad (17.5)$$

To ensure this we choose  $\kappa_\lambda = 0$ . You can think of this the other way around as well: pick  $\kappa_\lambda$  to be whatever you want, and that will fix  $\lambda$  to equal the three-point vertex at some values of the momenta.

## A Variational calculus

While one can memorize the rules for manipulating functions, functionals, and variational/functional derivatives, things can get hairy fast if we're not careful about the definitions. Most QFT textbooks are not careful about these definitions, although [2] is an exception (see Chapter 7).

We define a functional  $S$  as a map from domain a space of functions  $\mathcal{C}$  to codomain a set of numbers, say  $\mathbb{R}$ :

$$S : \mathcal{C} \rightarrow \mathbb{R}. \tag{A.1}$$

An example of a space of functions we will care about is field configurations over spacetime  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ . The argument of a functional is often written with square brackets,  $S[\phi]$ . Notice it inputs the *entire* field configuration, meaning the values  $\phi(t, \mathbf{x})$  for all  $t$  and  $\mathbf{x}$ . The action of a theory,  $S = \int dt L$ , is an example of such a functional. Assuming a fluid approximation to the air in this room, then the average temperature is another functional, which inputs the temperature  $T(\mathbf{x})$  for all  $\mathbf{x}$  and outputs a number.

Notice that for such a functional like  $S$ , it doesn't make sense to write  $S[\phi, \partial_\mu \phi]$ : since  $S$  already depends on  $\phi$  everywhere, it also knows about  $\partial_\mu \phi$  everywhere, so it is not an independent argument.

The Lagrangian is something you integrate against time to get the action, so we write

$$S[\phi] = \int_{-\infty}^{\infty} dt L[\phi(t), \dot{\phi}(t)]. \tag{A.2}$$

The dot denotes a time derivative. We will assume throughout these notes that Lagrangians depend only on powers of fields and their first derivatives. We have represented the Lagrangian as a functional of two functions. **The first function is the field configuration over all of space at a fixed time  $t$ , and the second function is the time derivative of the field configuration over all of space at the same time  $t$ . These two functions are independent.** So we have restricted the space of functions  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  to a new space of functions, which we will call  $\phi(t) : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ . It inputs a spatial point  $\mathbf{x}$  and outputs a number. This is useful notation: when a field which enters into a functional is missing some arguments, that means we consider the field configuration over all values of the missing arguments. With this notation,  $\phi(t, \mathbf{x})$  is not a function, it is instead the number obtained by evaluating the function  $\phi$  at  $t, \mathbf{x}$ , or evaluating the function  $\phi(t)$  at  $\mathbf{x}$ .

We can write the Lagrangian  $L$  as a spatial integral over the Lagrangian density  $\mathcal{L}$ :

$$S[\phi] = \int_{-\infty}^{\infty} dt L[\phi(t), \dot{\phi}(t)] = \int d^d x \mathcal{L}(\phi(t, \mathbf{x}), \partial\phi(t, \mathbf{x})/\partial x^\mu) = \int d^d x \mathcal{L}(\phi(x), \partial_\mu\phi(x)), \quad (\text{A.3})$$

where  $x$  is a spacetime point. While it looks like the Lagrangian density is not a Lorentz-invariant object because of the floating  $\mu$  index in the argument, any Lorentz-invariant Lagrangian will have this contracted, e.g. in a term like  $\partial_\mu\phi(x)\partial^\mu\phi(x)$ . The Lagrangian density is written with circular brackets instead of square brackets because it is not a functional, since it inputs two numbers (the values of the field and its derivative at the same spacetime point) and outputs a number. Considered over all spacetime it gives a function of space and time that can be integrated. We could generalize the Lagrangian density to depend on even higher derivatives, e.g.  $\partial_\mu\partial_\nu\phi(x)$ , but we will not do so.

We want to vary the functional  $S$  with respect to the field  $\phi$ . This means we want to make a change to our field  $\phi(x) \rightarrow \phi(\alpha, x)$ , where  $\alpha$  is a deformation parameter with  $\phi(0, x) = \phi(x)$ . For example we can change the field as  $\phi(x) \rightarrow \phi(\alpha, x) = e^\alpha\phi(x)$ . The linear change in the function is given by Taylor's theorem as  $\alpha\partial\phi(\alpha, x)/\partial\alpha|_{\alpha=0}$ . This is *defined* as  $\alpha\delta\phi$ ,

$$\delta\phi \equiv \left. \frac{\partial\phi(\alpha, x)}{\partial\alpha} \right|_{\alpha=0}. \quad (\text{A.4})$$

The action, Lagrangian, and Lagrangian density can all be thought of as functions of this new parameter  $\alpha$ , and therefore all have linear changes defined in the same way,

$$\delta S \equiv \left. \frac{\partial S[\phi(\alpha)]}{\partial\alpha} \right|_{\alpha=0}, \quad \delta L \equiv \left. \frac{\partial L[\phi(\alpha, t), \dot{\phi}(\alpha, t)]}{\partial\alpha} \right|_{\alpha=0}, \quad \delta \mathcal{L} \equiv \left. \frac{\partial \mathcal{L}(\phi(\alpha, x), \partial_\mu\phi(\alpha, x))}{\partial\alpha} \right|_{\alpha=0} \quad (\text{A.5})$$

For functionals of multiple fields, we could also define a variation like  $\Delta L = \left. \frac{\partial L[\phi(\alpha, t), \dot{\phi}(t)]}{\partial\alpha} \right|_{\alpha=0}$ , which varies  $\phi$  but not  $\dot{\phi}$ . The notation  $\delta L$  will almost always refer to the case where the variation is done on both fields in the same way, as it descends from a variation of the full spacetime field  $\phi$  in the action.

Varying the field  $\phi$  over all of spacetime tells us how its derivative  $\partial_\mu\phi$  varies,

$$\delta(\partial_\mu\phi) = \left. \frac{\partial(\partial_\mu\phi(\alpha, x))}{\partial\alpha} \right|_{\alpha=0} \quad (\text{A.6})$$

Notice that by this definition of  $\delta\phi$  we immediately have the identity

$$\delta(\partial_\mu\phi) = \left[ \frac{\partial(\partial_\mu\phi(\alpha, x))}{\partial\alpha} \right]_{\alpha=0} = \left[ \partial_\mu \frac{\partial\phi(\alpha, x)}{\partial\alpha} \right]_{\alpha=0} = \partial_\mu \left[ \frac{\partial\phi(\alpha, x)}{\partial\alpha} \right]_{\alpha=0} = \partial_\mu(\delta\phi) \quad (\text{A.7})$$

In the first equality we used commutativity of partial derivatives, in the second equality we used the fact that  $\phi(\alpha, x)$  admits a Taylor expansion in  $\alpha$ , and in the final equality we used the definition of  $\delta\phi$ .

Now let's impose stationarity of the action,  $\delta S = 0$ , and see what it implies about the Lagrangian density  $\mathcal{L}$ :

$$\delta S = \left[ \frac{\partial}{\partial \alpha} \int d^4x \mathcal{L}(\phi(\alpha, x), \partial_\mu \phi(\alpha, x)) \right]_{\alpha=0} = \left[ \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial(\partial_\mu \phi)}{\partial \alpha} \right) \right]_{\alpha=0} \quad (\text{A.8})$$

$$= \left[ \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \frac{\partial \phi}{\partial \alpha} \right]_{\alpha=0} = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi = 0, \quad (\text{A.9})$$

where we freely moved the derivative  $\partial/\partial\alpha$  past the integral and used the chain rule in the third expression, in the fourth expression we commuted the  $\alpha$  and  $\mu$  derivatives past each other and then integrated by parts (dropping boundary terms since we assume the variation  $\delta\phi$  goes to zero at the boundaries), and in the fifth expression we used the definition of  $\delta\phi$ . To reduce clutter we have dropped the arguments of  $\mathcal{L}$ , but notice in the fourth expression we are evaluating the derivatives at  $\phi(\alpha, x)$  whereas in the fifth expression we are evaluating them at  $\phi(x) := \phi(\alpha = 0, x)$ .

The vanishing of the above expression for arbitrary  $\delta\phi$  gives us the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}. \quad (\text{A.10})$$

In the main text of these notes, we will often mix notation and write  $\mathcal{L}(\phi, \partial_\mu \phi)$  instead of what should more appropriately be written as  $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$ . But the latter would lead to too much clutter. The fact that we use round brackets is what should remind you that it is a function of  $\phi(x)$  and  $\partial_\mu \phi(x)$ , and not a functional of  $\phi$  and  $\partial_\mu \phi$  (which as we already discussed would be a redundant dependence on  $\partial_\mu \phi$  for a functional).

## A.1 The functional derivative

The functional derivative is a derivative of a functional with respect to a function, the notation for which is  $\delta S/\delta\phi(x)$ . One way to define it is by the first order changes given above,

$$\int d^d x \frac{\delta S}{\delta\phi(x)} \delta\phi(x) := \left. \frac{\partial S[\phi(\alpha)]}{\partial \alpha} \right|_{\alpha=0} \quad (\text{A.11})$$

In particular this tells us

$$\delta S[\phi] = \int d^d x \frac{\delta S}{\delta \phi(x)} \delta \phi(x). \quad (\text{A.12})$$

If we imagine discretizing spacetime into a lattice of points  $x_i$  with  $i = 1, \dots, n$ , then  $S$  is an ordinary function of  $n$  variables  $\phi(x_i) \rightarrow x_i$  and this is just the chain rule

$$dS = \sum_{i=1}^n \frac{\partial S}{\partial x_i} dx_i. \quad (\text{A.13})$$

We haven't yet explained how to take functional derivatives in practice, but they work basically like partial derivatives, except with the discrete  $\rightarrow$  continuous replacement

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \longrightarrow \frac{\delta \phi(x)}{\delta \phi(y)} = \delta^d(x - y). \quad (\text{A.14})$$

Notice that both of these very standard way of writing derivatives misuse our notation. We are supposed to differentiate functions and functionally differentiate functionals. So  $x_i$  is not something one should partially differentiate. The proper way to write it is a bit ridiculous: we define a function  $f^i : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $f^i(x) = x^i$  and we differentiate  $\partial f^i / \partial x_j$ . Similarly, we should really write  $\mathcal{F}_x[\phi] = \phi(x)$  (this functional eats the full function  $\phi$  and evaluates it at  $x$ , it can be defined explicitly by  $\mathcal{F}_x[\phi] = \int d^d y \phi(y) \delta^d(x - y)$ ) and functionally differentiate  $\mathcal{F}_x$ . We will instead use the efficient notation of (A.14).

The identity  $\delta \phi(x) / \delta \phi(y) = \delta^d(x - y)$  can be derived by ensuring (A.12) is consistent when choosing  $S[\phi] = \int d^d x \phi(x)$ . For another way see the example below.

Another (equivalent) way to define functional derivatives is similar to how partial derivatives are defined. Say we have a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . We can take the partial derivative in the  $i$ 'th direction as

$$\frac{\partial f}{\partial x^i} := \lim_{\epsilon \rightarrow 0} \frac{f(x^1, \dots, x^i + \epsilon, \dots, x^n) - f(x^1, \dots, x^n)}{\epsilon} \quad (\text{A.15})$$

We can pick a basis for the domain  $\{\boldsymbol{\delta}_j\}$ ,  $j = 1, \dots, n$ , defined by  $(\boldsymbol{\delta}_j)^i = \delta_j^i$ . Then we can write this as

$$\frac{\partial f}{\partial x^i} = \lim_{\epsilon \rightarrow 0} \frac{f(\mathbf{x} + \epsilon \boldsymbol{\delta}_i) - f(\mathbf{x})}{\epsilon} \quad (\text{A.16})$$

To move to a derivative of a functional, we now we imagine taking this discrete set of  $n$  points to a continuous infinity of points. As before that means our set of variables  $\{x_j\}$  goes to a

function of a continuous variable  $\phi(x)$ . The  $i$ 'th direction becomes a particular spacetime point  $y$ . And our function  $f$  becomes a functional we will call  $S$ . The functional derivative is defined as

$$\frac{\delta S[\phi]}{\delta \phi(y)} := \lim_{\epsilon \rightarrow 0} \frac{S[\phi + \epsilon \delta_y] - S[\phi]}{\epsilon} \quad (\text{A.17})$$

We have defined a function  $\delta_y : \mathbb{R}^d \rightarrow \mathbb{R}$ , which is a Dirac delta function peak at  $y$ . This is *not*  $\delta(y)$ : instead, it inputs a point in spacetime, say  $x$ , and outputs  $\delta(y - x)$ . This is the continuum version of  $\delta_i$ , which takes in a discrete index  $j$  and spits out the Kronecker delta  $\delta_i^j = j$ . So  $S[\phi + \epsilon \delta_y]$  is defined such that you feed the functional  $S$  the field  $\phi$  at all spacetime points  $x \neq y$ , and for  $x = y$  you feed it  $\phi(y) + \delta(0)$ .

*Example 1:* Let's derive  $\delta\phi(x_0)/\delta\phi(y) = \delta^d(x_0 - y)$  via (A.17). As we discussed below (A.14) this is imprecise notation, since  $\phi(x_0)$  is not a functional. But we can associate a functional  $\Phi_{x_0}$  to functions evaluated at a point  $x_0$ . It inputs a function and outputs the value of that function at  $x_0$ . A precise formula is  $\Phi_{x_0} = \int d^d x \phi(x) \delta(x - x_0)$ . We have

$$\frac{\delta \Phi_{x_0}}{\delta \phi(y)} = \lim_{\epsilon \rightarrow 0} \frac{\Phi_{x_0}[\phi + \epsilon \delta_y] - \Phi_{x_0}[\phi]}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\phi(x_0) + \epsilon \delta^d(x_0 - y) - \phi(x_0)}{\epsilon} = \delta^d(x_0 - y). \quad (\text{A.18})$$

*Example 2:* Let's repeat the exercise of imposing stationarity of the action,  $\delta S = 0$ , but now study its implication for the Lagrangian instead of the Lagrangian density. We need a slight generalization of (A.12), since our functional now has two arguments instead of one. The generalization is

$$\left. \frac{\partial L[\phi(\alpha, t), \dot{\phi}(\alpha, t)]}{\partial \alpha} \right|_{\alpha=0} = \int d^d x \left( \frac{\delta L}{\delta \phi(x)} \delta \phi(x) + \frac{\delta L}{\delta (\partial_\mu \phi(x))} \delta (\partial_\mu \phi(x)) \right) \quad (\text{A.19})$$

This is just the chain rule. With the rest of our definitions we now have

$$\left[ \frac{\partial}{\partial \alpha} \int dt L[\phi(\alpha, t), \dot{\phi}(\alpha, t)] \right]_{\alpha=0} = \left[ \int dt \left( \frac{\delta L}{\delta \phi} \frac{\partial \phi}{\partial \alpha} + \frac{\delta L}{\delta \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \alpha} \right) \right]_{\alpha=0} \quad (\text{A.20})$$

$$= \left[ \int dt \left( \frac{\delta L}{\delta \phi} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}} \right) \frac{\partial \phi}{\partial \alpha} \right]_{\alpha=0} = \int dt \left( \frac{\delta L}{\delta \phi} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}} \right) \delta \phi = 0, \quad (\text{A.21})$$

using the same tricks as before. For arbitrary variation  $\delta\phi$  this therefore tells us

$$\frac{\delta L}{\delta\phi} = \frac{d}{dt} \frac{\delta L}{\delta\dot{\phi}} \quad (\text{A.22})$$

## B Fourier transforms

The Fourier transform is defined as

$$f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k}), \quad (\text{B.1})$$

with the inverse transform

$$\tilde{f}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}). \quad (\text{B.2})$$

We will often just call both the Fourier transform, and put the tilde on whichever is convenient.

The Fourier transform of 1 is a delta function,

$$\delta^3(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad 1 = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \delta^3(\mathbf{x}). \quad (\text{B.3})$$

This is necessary for the inverse transform to be an inverse:

$$\tilde{f}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3k'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{x}} \tilde{f}(\mathbf{k}') = \int d^3k' \delta^3(\mathbf{k} - \mathbf{k}') \tilde{f}(\mathbf{k}') = \tilde{f}(\mathbf{k}). \quad (\text{B.4})$$

The delta function Fourier transform is very useful, since it lets us write

$$\nabla^{2n} \delta^3(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \nabla^{2n} e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3k}{(2\pi)^3} (-k^2)^n e^{i\mathbf{k}\cdot\mathbf{x}} \quad (\text{B.5})$$

where  $\nabla^2 = \partial_i \partial^i$  is the Laplacian operator and  $k^2 = \mathbf{k} \cdot \mathbf{k} = k_i k^i$ . So we have

$$\widetilde{\nabla^{2n} \delta^3} = (-k^2)^n. \quad (\text{B.6})$$

For relativistic theories, we have

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik_\mu x^\mu} \tilde{f}(k), \quad \tilde{f}(k) = \int d^4 x e^{-ik_\mu x^\mu} f(x). \quad (\text{B.7})$$

So for example

$$\delta^4(x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik_\mu x^\mu}, \quad (\text{B.8})$$

$$\square^n \delta^4(x) = \int \frac{d^4 k}{(2\pi)^4} \square^n e^{ik_\mu x^\mu} = \int \frac{d^4 k}{(2\pi)^4} (-k^2)^n e^{ik_\mu x^\mu}. \quad (\text{B.9})$$

where  $\square = \partial_\mu \partial^\mu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$ . More generally this gives

$$\square^n f(x) = \int \frac{d^4 k}{(2\pi)^4} \square^n \tilde{f}(k) e^{ik_\mu x^\mu} = \int \frac{d^4 k}{(2\pi)^4} (-k^2)^n \tilde{f}(k) e^{ik_\mu x^\mu} \quad (\text{B.10})$$

The shorthand relations

$$\nabla^2 \leftrightarrow -\mathbf{k}^2, \quad \square \leftrightarrow -k^2 \quad (\text{B.11})$$

will come in very handy.

## C Green's functions

Say we want to solve an inhomogeneous ordinary differential equation like

$$y''(t) + \omega^2 y(t) = f(t) \quad (\text{C.1})$$

with source  $f(t)$ . This equation describes a driven harmonic oscillator, with driving force (divided by mass) given by  $f(t)$ . To solve (C.1) we instead first solve for the **Green's function**, defined by

$$\frac{d^2}{dt^2} G(t, t') + \omega^2 G(t, t') = \delta(t' - t). \quad (\text{C.2})$$

So the Green's function solves the same differential equation except the source term is chosen to be a delta function. With this solution in hand, we can construct a solution to (C.1):

$$y_p(t) = \int_{-\infty}^{\infty} G(t, t') f(t') dt'. \quad (\text{C.3})$$

The subscript  $p$  is to denote that this is a “particular” solution to the inhomogeneous differential equation. It is “particular” because it has no arbitrary constants (two are needed to fully fix this second order equation). To check that it is a solution simply plug into (C.1). This Green's function technique be generalized to equations of the form  $y'' + p(t)y' + q(t)y = f(t)$ . A solution to (C.1) is also given by

$$y(t) = \int_a^b G(t, t') f(t') dt' \quad (\text{C.4})$$

as long as  $t \in (a, b)$ . What we have done is invert the differential operator  $d^2/dt^2 + \omega^2$ , which gives us an integral operator with the kernel of the integral being the Green's function. Notice that to invert a differential operator there must be no vanishing eigenvalues (just like with inverting a matrix) – we assumed this to be the case above, and whether it is actually true in an example depends very much on the chosen boundary conditions.

Notice that we can always add  $y_h(t)$ , which solves the homogeneous equation  $y''(t) + \omega^2 y(t) = 0$ , to our constructed solution (C.3), obtaining

$$y(t) = y_h(t) + y_p(t). \quad (\text{C.5})$$

This will continue to solve (C.1).

What about boundary conditions? This is related to the freedom in arbitrary constants. Let's consider the physical situation where the source  $f(t)$  vanishes for  $t \leq t_0$ , and we turn it on for all  $t > t_0$ . In this case we expect  $y(t_0) = y_h(t_0)$  for  $t \leq t_0$ . Let's say our boundary conditions are  $y(t_0) = y_0$ ,  $y'(t_0) = v_0$ . We will put these boundary conditions into  $y_h(t)$ , so we have

$$y(t_0) = y_h(t_0) + y_p(t_0) = y_0 + 0, \quad y'(t_0) = y'_h(t_0) + y'_p(t_0) = v_0 + 0, \quad (\text{C.6})$$

i.e.  $y_p(t_0) = y'_p(t_0) = 0$ .  $G(t, t')$  inherits this boundary condition as  $G(t, t') = \partial_t G(t, t') = 0$  for  $t \leq t_0$ . So altogether we have

$$y''(t) + \omega^2 y(t) = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0 \quad (\text{C.7})$$

solved by

$$y(t) = y_h(t) + \int_{-\infty}^{\infty} G(t, t') f(t') dt' = y_h(t) + \int_{t_0}^{\infty} G(t, t') f(t') dt', \quad (\text{C.8})$$

$$G(t, t')|_{t \leq t_0} = \partial_t G(t, t')|_{t \leq t_0} = 0, \quad y_h''(t) + \omega^2 y_h(t) = 0, \quad y_h(t_0) = y_0, \quad y_h'(t_0) = v_0. \quad (\text{C.9})$$

We can check again by plugging in that this solves the differential equation and boundary conditions (C.7). But if we are thinking of physical time evolution, this is a little strange. To get  $y(t)$  at time  $t$ , we need to do an integral over all times to the future of  $t$ , since  $t' \in (t_0, \infty)$ ! Does the function  $y(t)$  actually depend on what happens to the future of the time we care about?!

To see that it doesn't, we should actually solve for  $G(t, t')$ . To find the solution, Laplace transform both sides of (C.1) to get

$$(p^2 + \omega^2)G(p, t') = e^{-pt'} \implies G(p, t') = \frac{e^{-pt'}}{p^2 + \omega^2}. \quad (\text{C.10})$$

Performing the inverse Laplace transform gives (recall we are considering  $t > t_0$ )

$$G(t, t') = \begin{cases} \frac{\sin \omega(t-t')}{\omega}, & t' < t \\ 0, & t < t'. \end{cases} \quad (\text{C.11})$$

The Green's function vanishes for  $t' > t$ , so times to the future of the time we care about do not influence  $y(t)$ ! Such a Green's function is known as a “retarded” Green's function  $G_R(t, t')$  and satisfies  $G_R(t, t' > t) = 0$ . This was assured by the boundary conditions  $G(t, t')|_{t \leq t_0} = \partial_t G(t, t')|_{t \leq t_0} = 0$ . So this is the relevant object when we have some initial field configuration (and its derivative) and want to propagate it into the future.

We can also define an “advanced” Green function  $G_A(t, t')$  which instead satisfies  $G_A(t, t' < t) = 0$ . An advanced Green function describes propagation into the past. It is useful if you know some final field configuration (and its derivative) and wanted to figure out where it came from. So we would have automatically produced an advanced Green function if we fixed boundary conditions  $y(t_0) = y_0, y'(t_0) = v_0$  and asked for  $y(t)$  for  $t < t_0$ .

Notice that the retarded and advanced Green's functions are not symmetric in  $t, t'$ . Physically this makes sense, but mathematically it may seem surprising since if our differential operator  $L$  is self-adjoint then it should assure that the Green's function is symmetric in its arguments (this is the reciprocity theorem or symmetry theorem; see e.g. [here](#) or Sturm-Liouville theory for more). But whether the operator is self-adjoint depends on the boundary

conditions. Self-adjoint is in reference to the inner product

$$\langle v|u\rangle = \int_a^b v^*(t)u(t)dt, \quad (\text{C.12})$$

for functions  $u, v \in \mathcal{L}^2([a, b])$ , i.e. square-integrable functions over  $[a, b]$ . So we have

$$\langle v, L[u]\rangle - \langle L[v], u\rangle = \int_a^b (v^*(u'' + \omega^2 u) - ((v^*)'' + \omega^2 v^*)u) dt \quad (\text{C.13})$$

$$= [v^*u' - (v^*)'u]_a^b \quad (\text{C.14})$$

This needs to vanish for arbitrary  $v, u$  for  $L$  to be self-adjoint. Notice that boundary conditions that set the function and its derivative to vanish at  $a$  (in the retarded case) or  $b$  (in the advanced case) do not lead to the above vanishing, i.e.  $L$  is not a self-adjoint operator with those boundary conditions. Notice instead that general Robin conditions  $\alpha f(b) + \beta f'(b) = 0$  and  $\gamma f(a) + \eta f'(a) = 0$  makes  $L$  self-adjoint. This includes, for example, Dirichlet boundary conditions at the two endpoints  $f(a) = f(b) = 0$ , or Neumann boundary conditions at the two endpoints  $f'(a) = f'(b) = 0$ .

Returning to our solution above, we note that it is sometimes also written as

$$y(t) = y_h(t) + \int_{t_0}^t G_R(t, t')f(t')dt', \quad (\text{C.15})$$

although this makes it less clear that (C.1) is satisfied.

*Example 1:* We would like to investigate the dependence on boundary conditions of the invertibility of differential operators. As a simple example that will be relevant for QFT, consider the equation

$$\left(\frac{d^2}{dt^2} + \omega^2\right)y(t) = f(t). \quad (\text{C.16})$$

To implement the Green's function method we want to solve

$$\left(\frac{d^2}{dt^2} + \omega^2\right)G(t, t') = \delta(t - t') \quad (\text{C.17})$$

by inverting the differential operator. This requires that it has no vanishing eigenvalues. Let's check this for some simple setups. In situations where we turn on the force  $f(t)$  from, say  $t = 0$  to  $t = 1$ , and we want to propagate the solution at  $t = 0$  forward in time, then we

saw that we want to solve (C.16) for  $y_p(t)$  with boundary conditions  $y_p(0) = y'_p(0) = 0$ . This will give the retarded propagator. So we want to see if there are any nontrivial  $y_p$  satisfying these boundary conditions and

$$\left(\frac{d^2}{dt^2} + \omega^2\right) y_p(t) = 0. \quad (\text{C.18})$$

We know what the general solution to this equation is:  $y_p(t) = Ae^{i\omega t} + Be^{-i\omega t}$  (in the case of  $y \in \mathbb{R}$  we would have  $B = A^*$ ). Implementing the boundary conditions gives

$$y_p(0) = 0 \implies A = -B, \quad y'_p(0) = 0 \implies A = B. \quad (\text{C.19})$$

This means  $y_p(t) = 0$ , i.e. there is no nontrivial solution with vanishing eigenvalue. So the retarded propagator can be constructed by inverting this differential operator. The advanced propagator works similarly: placing boundary conditions  $y_p(1) = y'_p(1) = 0$  also leads to a vanishing function  $y_p(t) = 0$ . The Feynman propagator has more interesting boundary conditions. If we define

$$y_+(t) = Ae^{i\omega t}, \quad y_-(t) = Be^{-i\omega t} \quad (\text{C.20})$$

then we can write  $y(t) = y_+(t) + y_-(t)$ . The Feynman boundary conditions place the retarded boundary conditions on  $y_+$  (which, in the quantum-mechanical context, are the Fourier modes that come with the creation operator  $a^\dagger$ ) and the advanced boundary conditions on  $y_-$  (which are the Fourier modes that come with the annihilation operator  $a$ ). In other words, we have

$$y_+(0) = y'_+(0) = 0, \quad y_-(1) = y'_-(1) = 0. \quad (\text{C.21})$$

But notice immediately that  $y_+(0) = 0 \implies A = 0$  and  $y_-(1) = 0 \implies B = 0$ . So there are again no vanishing eigenvalues.

We can use the Green's function method for partial differential equations as well. The case of Poisson's equation

$$\nabla^2 V = 4\pi\rho(\mathbf{r}) = 4\pi\rho(x, y, z) \quad (\text{C.22})$$

gives a nice physical picture of the method. This is the equation governing the electric (or gravitational) potential  $V$ , determined in terms of the charge (or mass) density  $\rho$ . We instead

study the Green's function equation

$$\nabla_{\mathbf{r}}^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') := \delta(x - x')\delta(y - y')\delta(z - z'), \quad (\text{C.23})$$

where the subscript is to remind us that the Laplacian acts on the first argument of the Green's function. We have replaced our continuous density with a point charge or point mass. To solve this equation, let's pick  $\mathbf{r}' = \mathbf{0}$  for simplicity, reintroducing it by a translation at the end. The equation we want to solve is

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} G(r) \right) = \delta(\mathbf{r}). \quad (\text{C.24})$$

We integrate both sides over a sphere of arbitrary radius  $R$  to get

$$\int_0^R \int_0^\pi \int_0^{2\pi} dr d\theta d\phi \frac{d}{dr} \left( r^2 \frac{d}{dr} G(r) \right) = 4\pi R^2 G'(R) = 1, \quad (\text{C.25})$$

where we used the divergence theorem. Since the sphere radius is arbitrary, we have

$$G'(r) = \frac{1}{4\pi r^2} \implies G(r) = -\frac{1}{4\pi r} \quad (\text{C.26})$$

Notice this solution is also consistent with  $\mathbf{r} \neq \mathbf{0}$ , since in that case  $\frac{1}{r^2} \frac{d}{dr} (r^2 G'(r)) = 0$ .<sup>25</sup> The general solution is therefore

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}. \quad (\text{C.27})$$

The solution to Poisson's equation is then given by

$$V(\mathbf{r}) = \int \int \int G(\mathbf{r}, \mathbf{r}') 4\pi \rho(\mathbf{r}') dV' = - \int \int \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (\text{C.28})$$

So the potential is obtained by integrating up the density. This makes sense: we are just approximating a continuous density by a bunch of point charges, and adding up the answers. That is the Green's function method.

The relativistic generalization of this is straightforward. We want to solve the equation

$$\square f = (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2) f = \rho. \quad (\text{C.29})$$

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<sup>25</sup>A simple way to get the scaling of  $G(r)$  is by considering  $r \neq 0$ ; to fix the coefficient we have to integrate over the delta function to pick up its strength.

The solution of this is sometimes written formally as

$$f = \frac{1}{\square} \rho \tag{C.30}$$

where  $\frac{1}{\square} = \square^{-1}$  is the inverse of the box operator. The inverse of a differential operator is an integral operator, and the kernel of the integral operator is the Green's function, as we just saw.<sup>26</sup> The reason we write  $\frac{1}{\square} \rho$  and not  $\frac{\rho}{\square}$  is to remember that  $\square^{-1}$  is an integral operator which will act on  $\rho$ . This notation looks crazy but it's meant to simplify computations. For example, let's say we wanted to solve the following differential equation

$$\square \phi + \lambda \phi^2 + J = 0 \tag{C.31}$$

perturbatively in the coefficient  $\lambda$ . This equation of motion follows from the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{3} \lambda \phi^3 + J \phi. \tag{C.32}$$

The first term is the kinetic term, the second term is an interaction term which makes the EOM nonlinear, and the third term is a “source” term which we will discuss more in the main text.

We first solve for the background configuration  $\phi_0$ , i.e. the one with  $\lambda = 0$ :

$$\square \phi_0 = -J \implies \phi_0 = -\frac{1}{\square} J. \tag{C.33}$$

We then look for the first correction  $\phi_1$ , which we parameterize to be  $\mathcal{O}(\lambda^0)$ :

$$\phi = \phi_0 + \lambda \phi_1 + \mathcal{O}(\lambda^2) \implies \square(\phi_0 + \lambda \phi_1) + \lambda(\phi_0 + \lambda \phi_1)^2 + J + \mathcal{O}(\lambda^2) = 0. \tag{C.34}$$

Using (C.33) gives

$$\square \phi_1 + \phi_0^2 + \mathcal{O}(\lambda^2) = 0. \tag{C.35}$$

Since we required  $\phi_1 \sim \mathcal{O}(\lambda^0)$ , we have

$$\phi_1 = -\frac{1}{\square} \phi_0^2 = -\frac{1}{\square} \left[ \left( \frac{1}{\square} J \right) \left( \frac{1}{\square} J \right) \right]. \tag{C.36}$$

---

<sup>26</sup>Very confusingly, sometimes  $\square^{-1}$  is called the Green's function, since  $\square G(x, y) = -\delta^4(x - y)$ , see below.

So the solution to order  $\lambda$  is given by

$$\phi = \phi_0 + \lambda\phi_1 = -\frac{1}{\square}J - \lambda\frac{1}{\square}\left[\left(\frac{1}{\square}J\right)\left(\frac{1}{\square}J\right)\right] + \mathcal{O}(\lambda^2). \quad (\text{C.37})$$

One can continue solving the equation to higher orders in this way. The fundamental object is the Green's function  $G(x, y)$ , alternatively called the propagator or 2-point function, which serves as the kernel for the integral operator  $-\square^{-1}$ . It is determined entirely by the quadratic terms in the Lagrangian, which in this case is just the kinetic term. It is called a propagator because it gives you the dynamics of how the field propagates once the interactions and source  $J$  are specified. Let's write (C.37) a little more explicitly.

The Green's function satisfies

$$\square_x G(x, y) = -\delta^4(x - y), \quad (\text{C.38})$$

where the minus sign on the RHS is due to the minus sign in (C.33), so this is the same convention as in (C.2). We solve this in Fourier space, i.e. we Fourier transform both sides with respect to  $x$ , using (B.10) to get the equation

$$\int \frac{d^4k}{(2\pi)^4} (-k^2) \tilde{G}(k, y) e^{ikx} = - \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \quad (\text{C.39})$$

which has as solution

$$\tilde{G}(k, y) = \frac{1}{k^2} e^{-iky} \implies G(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik(x-y)} = G(x - y). \quad (\text{C.40})$$

The Green's function only depends on the relativistic separation  $x-y := \sqrt{(x^\mu - y^\mu)(x_\mu - y_\mu)}$ , so we write  $G(x - y)$ . One can check by direct calculation that this satisfies (C.38).

We use the Green's function to construct the zeroeth order solution<sup>27</sup>  $\phi_0 = -\frac{1}{\square}J$

$$\phi_0 = \int d^4y G(x - y) J(y). \quad (\text{C.44})$$

---

<sup>27</sup>It is simplest to check this claimed solution by acting with  $\square$  on both sides. But if you want to construct it more explicitly, write

$$\square\phi_0(x) = -J(x) \implies - \int d^4x G(x - y) \square\phi_0(x) = \int d^4x G(x - y) J(x) \quad (\text{C.41})$$

$$\implies - \int d^4x (\square G(x - y)) \phi_0(x) = \int d^4x G(x - y) J(x) \implies \int d^4x \delta^4(x - y) \phi_0(x) = \int d^4x G(x - y) J(x) \quad (\text{C.42})$$

$$\implies (\text{C.44}) \quad (\text{C.43})$$

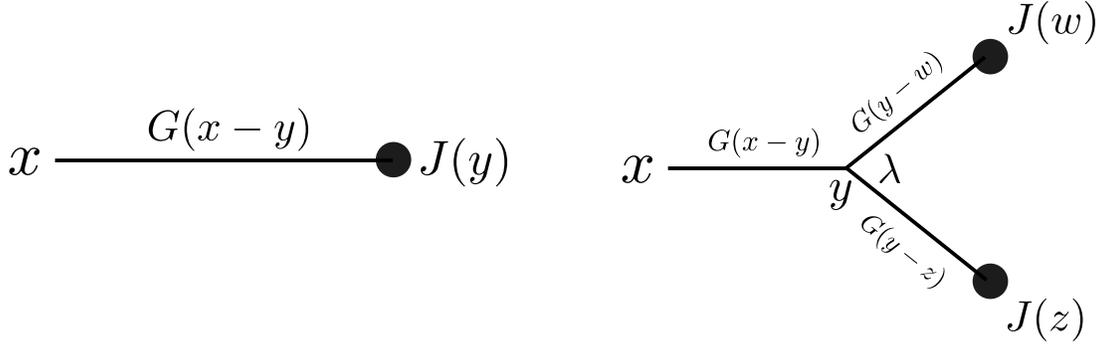
To go to the next order we need to use (C.35):

$$\square\phi_1(y) = -\phi_0^2(y) = -\int d^4w G(y-w)J(w) \int d^4z G(y-z)J(z). \quad (\text{C.45})$$

We can identify  $-\phi_0^2(y) = -J(y)$  as a source so that we can use again the Green's function solution  $\phi_1 = \int d^4y G(x-y)J(y) = \int d^4y G(x-y)\phi_0^2(y)$ . We combine this with  $\phi_0$  to write  $\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \mathcal{O}(\lambda^2)$  as

$$\begin{aligned} \phi(x) = & \int d^4y G(x-y)J(y) \\ & + \lambda \int d^4y G(x-y) \int d^4w G(y-w)J(w) \int d^4z G(y-z)J(z) + \mathcal{O}(\lambda^2). \end{aligned} \quad (\text{C.46})$$

We can think of the sources  $J$  as insertions, which are then propagated some distance via  $G(x-y)$ . In the first line we simply have an insertion  $J(y)$  which is propagated via  $G(x-y)$  to  $x$ . The point  $y$  needs to be integrated over all spacetime. In the second line we have two insertions,  $J(w)$  and  $J(z)$ , and they are both propagated and meet at the point  $y$ , which is then propagated to the point  $x$ . There are integrals over  $w$  and  $z$  so these insertions have to be integrated over all spacetime. The point  $y$  is also integrated over all spacetime but there is no insertion there, it is an ‘‘interaction vertex’’ which comes with a cost of  $\lambda$ . This is how this diagram represents the  $\mathcal{O}(\lambda)$  correction to the field value  $\phi(x)$ . One can pictorially represent this solution with the Feynman diagrams below.



The Lagrangian we have written down in fact describes gravitational interactions, with  $\phi$  corresponding to (a piece of) the graviton. As discussed in [4], this expansion lets us discuss the way the sun affects the Earth. The source is the sun, and the source function  $J$  will only have support on the location of the sun (i.e. the integrals over spacetime will only have contributions from the part of space where the sun is). The leading term  $\phi_0$  represents the Newtonian potential, since that results in Poisson's equation, which is linear. The nonlinear

corrections come from Einstein's theory. The first one is represented by  $\phi_1$ . But further corrections can be drawn, according to the pictorial rules of Feynman diagrams. To compute to  $\mathcal{O}(\lambda^n)$ , we follow the steps:

- Draw a point  $x$  and a line from  $x$  to a new point  $x_i$ .
- Either truncate the line at  $x_i$  with a source  $J$ , or draw two lines coming out of  $x_i$ , at a cost of one factor of  $\lambda$ . If you draw two lines, they need to truncate at sources or branch further. Repeat this until you have paid  $m$  costs of  $\lambda$ , with  $m \leq n$ .
- Repeat this for all integer  $0 \leq m \leq n$ .
- The final value of  $\phi(x)$  is given by summing all the graphs constructed above. The picture is translated to equations by replacing lines between  $x_i$  and  $x_j$  with propagators  $G(x_i - x_j)$  and integrating over all  $x_i$  (but not  $x$ ).

Notice that according to these rules the number of factors of  $\lambda$  in a diagram will always be one less than the number of factors of  $J$ .

Different classical theories have a similar Feynman diagrammatic expansion, with the primary difference being the possibility of different species of particles and higher-order interactions, which correspond to more lines coming out of each branching point. Quantum theories have a very similar diagrammatic expansion, except we allow the possibility for lines to close in on themselves.

## D Gaussian integrals

### D.1 One-dimensional integrals

Let's calculate

$$\mathcal{I} = \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2 + Jp}. \quad (\text{D.1})$$

We want to get rid of the linear-in- $p$  piece. This is done by "completing the square"

$$\mathcal{I} = \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}a(p - \frac{J}{a})^2 + \frac{J^2}{2a}} \quad (\text{D.2})$$

and then shifting  $p \rightarrow p + J/a$ , which does not change the integration measure:

$$\mathcal{I} = e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2} = \frac{1}{\sqrt{a}} e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}p^2} \quad (\text{D.3})$$

We can evaluate this integral by using a trick:

$$\left(\int_{-\infty}^{\infty} dp e^{-\frac{1}{2}p^2}\right)^2 = \int dx \int dy e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} = \int_0^{\infty} r dr \int_0^{2\pi} d\phi e^{-\frac{1}{2}r^2} \quad (\text{D.4})$$

$$= 2\pi \int_0^{\infty} \frac{u du}{2} e^{-\frac{1}{2}u} = 2\pi. \quad (\text{D.5})$$

So we find the following formula:

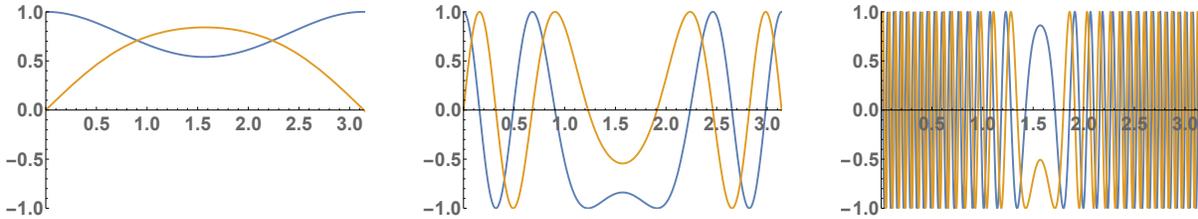
$$\boxed{\int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2 + Jp} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}.} \quad (\text{D.6})$$

### Stationary phase/saddle point/steepest descent

In the limit  $\hbar \rightarrow 0$ , integrals like

$$\int dp e^{-f(p)/\hbar}, \quad \text{or} \quad \int dp e^{if(p)/\hbar} \quad (\text{D.7})$$

tend to be dominated by the value of  $p$  for which  $f(p)$  is an extremum. This is easy to see in the former integral, where we have humongous exponential suppressions which we can fight by minimizing  $f(p)$ . But the latter integral is similar: as  $\hbar \rightarrow 0$  the phase is cycling around more and more quickly, i.e. a full cycle of the phase requires a smaller and smaller range of  $p$  as  $\hbar \rightarrow 0$ . These cycles integrate to zero, and the only way to stop the cycling is to make  $f(p)$  not change as  $p$  changes; this is just an extremum. The location of the extrema are called “saddle points.” As an example, we plot below the real and imaginary parts of  $e^{i \sin t/\hbar}$  for  $\hbar = 1, .1, .01$  from left to right. Notice that by  $\hbar = .01$  we have a scribbly mess which is going to cancel against itself except around  $\pi/2$ , which is the extremum of  $\sin t$  in the plotted range  $t \in (0, \pi)$ .



Up to the prefactor, we could have obtained formula (D.6) by applying the saddle point method, even though there isn't a manifest small  $\hbar$ . The saddle point  $p_*$  of the integrand of

(D.1) is given by

$$\left. \frac{d}{dp} \left( -\frac{1}{2}ap^2 + Jp \right) \right|_{p_\star} = 0 \implies -ap_\star + J = 0 \implies p_\star = J/a. \quad (\text{D.8})$$

The integral is then approximated by

$$\mathcal{I} \approx e^{(-\frac{1}{2}ap^2 + Jp)|_{p_\star}} = e^{\frac{J^2}{2a}}. \quad (\text{D.9})$$

This is a good approximation when  $J^2/a$  is large. To obtain the prefactor we would have to do a Gaussian integral (without source  $J$ ). This is done by expanding  $p = p_\star + \delta p$  and working to quadratic order in  $\delta p$ . We won't go through the details since we will often be uninterested in these prefactors.

## D.2 Multi-dimensional integrals

The above has a simple generalization to multi-dimensional Gaussian integrals, where we integrate over a vector  $p^i$  of variables:

$$\mathcal{I} = \int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2}A_{ij}p^i p^j + J_i p^i} = \int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2}p^T A p + J^T p} \quad (\text{D.10})$$

where  $A$  is a symmetric positive-definite matrix (which we can assume since  $p^i p^j$  is symmetric and so any antisymmetric piece of  $A$  vanishes in  $A_{ij}p^i p^j$ ). We first eliminate the source term by completing the square:

$$\mathcal{I} = \int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2}A_{ij}(p^i - (A^{-1})^{ik} J_k)(p^j - (A^{-1})^{jm} J_m) + \frac{1}{2}J_j J_m (A^{-1})^{mj}} \quad (\text{D.11})$$

$$= \int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2}(p - A^{-1}J)^T A (p - A^{-1}J) + \frac{1}{2}J^T A^{-1}J}. \quad (\text{D.12})$$

Performing the shift  $p \rightarrow p + A^{-1}J$  gives

$$\mathcal{I} = e^{\frac{1}{2}J^T A^{-1}J} \int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2}p^T A p}. \quad (\text{D.13})$$

We can write  $A = Q^T D Q$  for an orthogonal matrix  $Q$  and diagonal matrix  $D$ . We can induce this by the coordinate change  $p \rightarrow Qp$ . Since  $|\det[Q]| = 1$ , the Jacobian from the change-of-variables is trivial and we can evaluate the above with  $A \rightarrow D$ . This integral is

simply  $n$  copies of (D.6), which gives

$$\boxed{\int_{-\infty}^{\infty} d^n p e^{-\frac{1}{2} p^T A p + J^T p} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2} J^T A^{-1} J}.} \quad (\text{D.14})$$

We again could have gotten this result by a saddle-point analysis

$$\left. \frac{d}{dp^k} \left( -\frac{1}{2} A_{ij} p^i p^j + J_i p^i \right) \right|_{p_\star^k} = 0 \implies -\frac{1}{2} A_{kj} p_\star^j - \frac{1}{2} A_{ik} p_\star^i + J_k = 0 \implies A_{ki} p_\star^i = J_k. \quad (\text{D.15})$$

Multiplying by the inverse matrix  $(A^{-1})^{jk}$  on both sides gives

$$p_\star^j = (A^{-1})^{jk} J_k \implies p_\star = A^{-1} J. \quad (\text{D.16})$$

Plugging this into (D.10) gives

$$\mathcal{I} \approx e^{-\frac{1}{2} p^T A p + J^T p|_{p_\star}} = e^{\frac{1}{2} J^T A^{-1} J}. \quad (\text{D.17})$$

## E Symmetries and representations in quantum mechanics

The precise definition of a symmetry has been updated over the years. Heuristically, it is a change of perspective (e.g. the transformation of fields as we saw in the proof of Noether's theorem) that does not change the results of experiments. In the context of classical field theory we saw an example of this: Noether's theorem. The transformation of the fields was the change in perspective. The results of experiments remain unchanged in that context since the action – and therefore the equations of motion – was invariant under the transformation of the fields.

In quantum mechanics, we speak of vectors in Hilbert space. We will define a *ray*  $\underline{v}$  to be all vectors related to  $|v\rangle$  by  $e^{i\alpha}|v\rangle$  with  $\alpha \in \mathbb{R}$ , i.e.  $\underline{v} = \{e^{i\alpha}|v\rangle; \alpha \in \mathbb{R}\}$ . The state of a system is represented by a ray in Hilbert space  $\underline{v}$ . A change in perspective can lead to a different observer representing the state of the system by a different ray  $\underline{v}'$ . However, if this is a symmetry, the results of experiments should remain unchanged, for example the probability to transition into some other state  $\underline{w}$  should remain unchanged:

$$P(\underline{v} \rightarrow \underline{w}) = P(\underline{v}' \rightarrow \underline{w}'). \quad (\text{E.1})$$

Eugene Wigner proved that symmetry transformations which satisfy the above imply the existence of an operator  $U$  such that if  $|v\rangle \in \underline{v}$  then  $U|w\rangle \in \underline{w}$  with  $U$  either (a) unitary and

linear:

$$\langle Uv|Uw\rangle = \langle v|w\rangle, \quad U(a|v\rangle + b|w\rangle) = aU|v\rangle + bU|w\rangle \quad (\text{E.2})$$

or (b) antiunitary and antilinear:

$$\langle Uv|Uw\rangle = \langle v|w\rangle^*, \quad U(a|v\rangle + b|w\rangle) = a^*U|v\rangle + b^*U|w\rangle \quad (\text{E.3})$$

Notice that unitary operators have to be linear and antiunitary operators have to be antilinear, you cannot have an e.g. antiunitary and linear operator.

The adjoint of a linear operator  $L$  is defined as

$$\langle v|L^\dagger w\rangle := \langle Lv|w\rangle \quad (\text{E.4})$$

This does not work for an antilinear operator, since taking a sum of vectors  $v \rightarrow v_1 + v_2$  would lead to the LHS being linear in  $v$  while the RHS is antilinear in  $v$ . So for an antilinear operator we instead define the adjoint as

$$\langle v|A^\dagger w\rangle := \langle Lv|w\rangle^* = \langle w|Lv\rangle. \quad (\text{E.5})$$

These definitions together mean that

$$U^\dagger = U^{-1} \quad (\text{E.6})$$

for both unitary and antiunitary operators.

Antiunitary operators are exotic and will not be considered in this class, they often involve reversing the orientation of time. Continuous symmetries, like the one considered in (4.16), have a real parameter that can be continuously varied to make the symmetry transformation trivial. Such a trivial transformation is represented by the identity operator  $U = 1$ , which is linear and unitary. By continuity, then, the more general symmetry transformation will also be linear and unitary. This is a heuristic reason why most transformations considered, like rotations or Lorentz boosts, are linear and unitary.

We can fruitfully expand such linear and unitary operators close to the identity operator:

$$U = 1 + i\epsilon t \quad (\text{E.7})$$

with  $\epsilon$  real and infinitesimal. Since  $U$  is unitary, this means  $t$  is real and Hermitian.

The set of symmetry transformations forms a group. We consider the transformation on

rays:

$$T_1 : \underline{v} \rightarrow \underline{v}' , \quad T_2 : \underline{v}' \rightarrow \underline{v}'' \quad \Longrightarrow \quad T_2 T_1 : \underline{v} \rightarrow \underline{v}'' \quad (\text{E.8})$$

We also have inverse transformations  $T^{-1}$  and the identity transformation  $T = 1$ .

We now want to understand the (anti)unitary operators  $U(T)$  corresponding to these symmetry transformations  $T$ . There is an annoying complication in quantum mechanics which is the reason we distinguished between vectors and rays. That complication is the fact that the (anti)unitary operators  $U(T)$  act on vectors in the Hilbert space, whereas  $T$  above was acting on rays. Since  $T_1 : \underline{v} \rightarrow \underline{v}'$ , we want that  $U(T_1)|v\rangle \in \underline{v}'$  for  $|v\rangle \in \underline{v}$ . Similarly,  $U(T_2)|v'\rangle \in \underline{v}''$ . This means that

$$U(T_2)U(T_1)|v\rangle = e^{i\phi(T_2, T_1)}U(T_2 T_1)|v\rangle \quad (\text{E.9})$$

i.e. both vectors live in the same ray, and the phase distinguishing the states can depend on the symmetry transformations  $T_1, T_2$ . Ignoring the possibility of superselection sectors, the phase  $\phi$  is independent of the state  $|v\rangle$ . (For the proof see pg. 52 - 53 of [2]) This means that we can write

$$U(T_2)U(T_1) = e^{i\phi(T_2, T_1)}U(T_2 T_1) \quad (\text{E.10})$$

When  $\phi = 0$  this is called a representation of the group of symmetry transformations given by  $T$ . For nonzero  $\phi$  this is instead called a projective representation, which is just a representation that allows for a phase factor as written above.

## F Lorentz transformations

### F.1 Lorentz group and subgroups

One of the most important symmetries we will be concerned with is Lorentz invariance. This is the symmetry of special relativity, and it connects the physics between inertial frames of reference. For  $x^\mu$  the coordinates in one inertial frame, and  $x'^\mu$  the coordinates in another inertial frame, we must have the relation

$$\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu , \quad (\text{F.1})$$

which can also be written as

$$\eta_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \eta_{\rho\sigma}. \quad (\text{F.2})$$

This means that the interval  $-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$  is preserved between points in spacetime.

The general transformation between coordinates  $x^{\mu}$  and  $x'^{\mu}$  obeying (F.1) can be written as

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}, \quad (\text{F.3})$$

where  $a^{\mu}$  are arbitrary constants and  $\Lambda^{\mu}_{\nu}$  is a matrix of constants satisfying

$$\eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}. \quad (\text{F.4})$$

The set of Lorentz transformations forms a group: the product of two Lorentz transformations is another Lorentz transformation (check this!), there is an identity transformation given by  $\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu}$  which corresponds to no change in frame, and each transformation has an inverse. To construct the inverse, we write (F.4) as

$$\Lambda_{\nu}^{\rho} \Lambda^{\nu}_{\sigma} = \delta^{\rho}_{\sigma} \quad (\text{F.5})$$

The matrix inverse is defined as

$$(\Lambda^{-1})^{\rho}_{\nu} \Lambda^{\nu}_{\sigma} = \delta^{\rho}_{\sigma}, \quad (\text{F.6})$$

which lets us read off

$$(\Lambda^{-1})^{\rho}_{\nu} = \Lambda_{\nu}^{\rho}. \quad (\text{F.7})$$

We can now classify various subgroups of Lorentz transformations. The full set of transformations is known as the **inhomogeneous Lorentz group** or **Poincaré group**. If we set  $a^{\mu} = 0$ , then this is known as the **homogeneous Lorentz group**.

We can further classify the group of Lorentz transformations. (F.7) tells us that  $(\text{Det } \Lambda)^{-1} = \text{Det } \Lambda$  and so

$$\text{Det } \Lambda = \pm 1. \quad (\text{F.8})$$

Transformations with  $\text{Det } \Lambda = +1$  are called **proper**, while transformations with  $\text{Det } \Lambda = -1$  are called **improper**. The homogeneous and inhomogeneous Lorentz groups have proper subgroups, since  $\text{Det } \Lambda = +1$  is a property that is preserved under successive Lorentz transformations. (Conversely, there is no such thing as an improper subgroup.)

Finally, we have the **orthochronous** subgroup which consists of transformations with  $\Lambda^0_0 \geq +1$ . Restrictions on the 0-0 component can be obtained from (F.4), which tells us  $(\Lambda^0_0)^2 - \Lambda^i_0 \Lambda^i_0 = 1$ , so either  $\Lambda^0_0 \geq +1$  or  $\Lambda^0_0 \leq -1$ .

Now for some examples. It will help to begin with continuous Lorentz transformations which can be expanded about the identity. The identity transformation is (F.3) with  $\Lambda^\mu_\nu = \delta^\mu_\nu$  and  $a^\mu = 0$ . This means the symmetry transformations near the identity are (F.3) with

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \quad a^\mu = \epsilon^\mu, \quad (\text{F.9})$$

where  $\omega^\mu_\nu$  and  $\epsilon^\mu$  are infinitesimal. Since the identity transformation is proper and orthochronous, by continuity any transformation that can be expanded about the identity is proper and orthochronous. This includes the usual Lorentz boosts, spatial rotations, and spacetime translations you are used to.

We also have the discrete transformations of parity and time reversal. The parity transformation is

$$P^\mu_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (\text{F.10})$$

while the time-reversal transformation is

$$T^\mu_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \quad (\text{F.11})$$

These are discrete so cannot be expanded around the identity. Parity is orthochronous but improper, while time reversal is nonorthochronous and improper. They both square to the identity.

In the full, inhomogeneous Lorentz group, any transformation is either proper and orthochronous, or can be written as a product of an element of the proper orthochronous subgroup with one of the discrete transformations  $P$ ,  $F$ , or  $PF$ . For this reason, people

usually restrict attention to the proper orthochronous Lorentz transformations (the ones you are probably used to!) and supplement them with  $P$  and  $F$ .

## F.2 Representations on Hilbert space

Notice that (F.4) and (F.9) tells us that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (\text{F.12})$$

This means that for an infinitesimal Lorentz transformation (F.9), we can write the unitary operator which represents the transformation as

$$U(1 + \omega, \epsilon) = 1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - i\epsilon_{\mu}P^{\mu} + \dots \quad (\text{F.13})$$

with  $M^{\mu\nu} = -M^{\nu\mu}$ . The ellipses represent higher orders in the infinitesimals.  $M^{\mu\nu}$  and  $P^{\mu}$  are Hermitian operators independent of  $\epsilon$  and  $\omega$ . We will shortly see that  $P^{\mu}$  is the momentum four-vector operator and  $M^{23}$ ,  $M^{31}$ ,  $M^{12}$  are components of the angular momentum vector operator.

We want to see how these operators transform under Lorentz transformations. We consider the product

$$U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a), \quad (\text{F.14})$$

where  $\Lambda$  and  $a$  are parameters of an arbitrary Lorentz transformation. Using (E.10) with the phase set to zero (to do this we need to extend the Lorentz group to  $\text{SL}(2, C)$ , see Chapter 2.7 of [2]), we have

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2a_1 + a_2), \quad (\text{F.15})$$

which tells us

$$U(\Lambda^{-1}, -\Lambda^{-1}a)U(\Lambda, a) = U(1, 0) \implies U(\Lambda^{-1}, -\Lambda^{-1}a) = U(\Lambda, a)^{-1}. \quad (\text{F.16})$$

Therefore we can write our product (F.14) as

$$U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a) = U(\Lambda(1 + \omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a). \quad (\text{F.17})$$

Expanding both sides to first order in  $\omega$  and  $\epsilon$  gives

$$U(\Lambda, a) \left[ \frac{1}{2} \omega_{\rho\sigma} M^{\rho\sigma} - \epsilon_{\rho} P^{\rho} \right] U^{-1}(\Lambda, a) = \frac{1}{2} (\Lambda \omega \Lambda^{-1})_{\mu\nu} M^{\mu\nu} - (\Lambda \epsilon - \Lambda \omega \Lambda^{-1} a)_{\mu} P^{\mu}. \quad (\text{F.18})$$

Equating the coefficients of  $\omega_{\rho\sigma}$  and  $\epsilon_{\rho}$  on both sides gives

$$U(\Lambda, a) M^{\rho\sigma} U^{-1}(\Lambda, a) = \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} (M^{\mu\nu} - a^{\mu} P^{\nu} + a^{\nu} P^{\mu}) \quad (\text{F.19})$$

$$U(\Lambda, a) P^{\rho} U^{-1}(\Lambda, a) = \Lambda_{\mu}^{\rho} P^{\mu} \quad (\text{F.20})$$

Notice that  $P^{\mu}$  is translation-invariant while  $M^{\mu\nu}$  is not; this simply tells us that energy/momentum is independent of the origin, and for the components of  $M^{\mu\nu}$  corresponding to angular momentum it is telling us that the angular momentum depends on the origin. The homogeneous Lorentz transformations (i.e. with  $a^{\mu} = 0$ ) are telling us that  $J^{\mu\nu}$  transforms as a tensor and  $P^{\mu}$  as a vector.

Picking  $\Lambda$  and  $a$  in (F.19) - (F.20) to be  $\Lambda_{\nu}^{\mu} = \delta_{\nu}^{\mu} + \omega_{\nu}^{\mu}$  and  $a^{\mu} = \epsilon^{\mu}$  (where  $\omega$  and  $\epsilon$  need not have any relation to the ones considered previously), these equations become

$$i \left[ \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu} - \epsilon_{\mu} P^{\mu}, M^{\rho\sigma} \right] = \omega_{\mu}^{\rho} M^{\mu\sigma} + \omega_{\nu}^{\sigma} M^{\rho\nu} - \epsilon^{\rho} P^{\sigma} + \epsilon^{\sigma} P^{\rho}, \quad (\text{F.21})$$

$$i \left[ \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu} - \epsilon_{\mu} P^{\mu}, P^{\rho} \right] = \omega_{\mu}^{\rho} P^{\mu}. \quad (\text{F.22})$$

Again equating the coefficients of  $\omega_{\mu\nu}$  and  $\epsilon_{\mu}$  on both sides gives us the Lie algebra of the Poincaré group:

$$i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu}, \quad (\text{F.23})$$

$$i[P^{\mu}, M^{\rho\sigma}] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}, \quad (\text{F.24})$$

$$[P^{\mu}, P^{\rho}] = 0. \quad (\text{F.25})$$

Recall that in quantum mechanics operators which commute with the Hamiltonian  $H = P^0$  are conserved in time. Our algebra above indicates that the momentum three-vector

$$\mathbf{P} = \{P^1, P^2, P^3\} \quad (\text{F.26})$$

and the angular momentum three-vector

$$\mathbf{J} = \{M^{23}, M^{31}, M^{12}\} \quad (\text{F.27})$$

all commute with  $H$ . And of course  $H$  commutes with  $H$ . The remaining generators form the boost three-vector

$$\mathbf{K} = \{M^{01}, M^{02}, M^{03}\} \quad (\text{F.28})$$

and do not commute with  $H$ . So our Lie algebra can be written as

$$[J_i, J_j] = i\epsilon_{ijk}J^k, \quad (\text{F.29})$$

$$[J_i, K_j] = i\epsilon_{ijk}K^k, \quad (\text{F.30})$$

$$[K_i, K_j] = -i\epsilon_{ijk}J^k, \quad (\text{F.31})$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k, \quad (\text{F.32})$$

$$[K_i, P_j] = -iH\delta_{ij}, \quad (\text{F.33})$$

$$[J_i, H] = [P_i, H] = [H, H] = 0, \quad (\text{F.34})$$

$$[K_i, H] = -iP_i. \quad (\text{F.35})$$

If we restrict to the homogeneous Lorentz group – which eliminates translations and therefore  $P^\mu$  – we have instead

$$[J_i, J_j] = i\epsilon_{ijk}J^k, \quad [J_i, K_j] = i\epsilon_{ijk}K^k, \quad [K_i, K_j] = -i\epsilon_{ijk}J^k. \quad (\text{F.36})$$

This lets us define two decoupled three-vectors

$$J_i^+ = \frac{1}{2}(J_i + iK_i), \quad J_i^- = \frac{1}{2}(J_i - iK_i) \quad (\text{F.37})$$

which obey the angular momentum algebra

$$[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+ \quad (\text{F.38})$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^- \quad (\text{F.39})$$

$$[J_i^+, J_j^-] = 0. \quad (\text{F.40})$$

This basis is useful since it allows us to leverage results from the study of angular momentum in quantum mechanics. For example, finding representations of this algebra becomes the same

problem as finding matrices representing the spins of two uncoupled particles.

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